Real Analysis Preliminary Examination

May, 2024

Directions: **Complete seven (7) of the following nine problems, and indicate in the box below which seven problems should be graded.**

If you do not do this, then problems 1-7 will be graded. Strive for clear, detailed, and legible solutions. Notations: $(X, \mathcal{M}, \mu)$ denotes a measure space, m denotes the Lebesgue measure on $\mathbb{R}$, and m_n denotes the Lebesgue measure on $\mathbb{R}^n$.

1. Prove that $\forall A \subset \mathbb{R}$ such that $m^*(A) < \infty$, $\exists$ a Borel set $B \subset \mathbb{R}$ such that $B \supset A$ and $m(B) = m^*(A)$.

2. Let $f, g : X \to [-\infty, \infty]$ such that $f = g$ a.e. If μ is complete and if f is $\mathcal{M}$-measurable, prove that g is also $\mathcal{M}$-measurable.

3. Let $f_k = \dfrac{(1-x)^k \cos \frac{k}{x}}{\sqrt{x}}$. Prove that $\displaystyle\lim_{k\to\infty} \int_{(0,1)} f_k(x)\, dm = 0$.

4. Suppose $\{f_n\} \subset L^+$, $f_n(x) \to f(x)$ $\forall x$, and $\displaystyle\int_X f = \lim_{n\to\infty} \int_X f_n < \infty$. Prove that
$$\lim_{n\to\infty} \int_E f_n = \int_E f \quad \forall E \in \mathcal{M}.$$

5. Let λ be a signed measure on $(X, \mathcal{M})$ and $\lambda = \lambda^+ - \lambda^-$ be its Jordan decomposition. Prove that
$$\lambda^+(E) = \sup\{\lambda(F) : F \subset E, F \in \mathcal{M}\},$$
and
$$\lambda^-(E) = -\inf\{\lambda(F) : F \subset E, F \in \mathcal{M}\}.$$

6. Let A be an open subset of $\mathbb{R}^2$. For an $h > 0$, define
$$B_h = \{(x, y, h) \in \mathbb{R}^3 : (x, y) \in A\}.$$
Define
$$C = \{((\lambda x, \lambda y, \lambda z) : (x, y, z) \in B_h, 0 \le \lambda \le 1\}.$$
Find $m_3(C))$ and prove your answer (*you need not prove that C is measurable*).

7. Let X, Y be Banach spaces. If $\{T_n\} \subset L(X, Y)$ (bounded linear operators) and $T_n \to T$ weakly, prove that $\sup_n \|T_n\| < \infty$.

8. If $1 \le p < q < r \le \infty$ and $f \in L^p \cap L^r$, prove that $f \in L^q$.

9. Let f, g be Lebesgue measurable functions on $[0, 1]$ such that $f(x), g(x) > 0$ $\forall x \in [0, 1]$, and ν and μ be measures (*you do not need to prove it*) defined by
$$\nu(E) = \int_E f\, dm, \quad \mu(E) = \int_E g\, dm, \quad \forall \text{ Lebesgue measurable } E \subset [0, 1].$$
Show that $\nu \ll \mu$, and determine the Radon-Nikodym derivative $\dfrac{d\nu}{d\mu}$ (*Hint: the Chain Rule*).