

Topology Preliminary Exam

August 2022

Directions

Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any major theorem is used in any argument, give a precise statement of that theorem.

There are eight questions on the exam. You will be graded on your seven best answers.

Problems

1. Consider the real line \mathbb{R} and its quotient \mathbb{R}/\mathbb{Q} by the subgroup \mathbb{Q} of rational numbers:

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$

Prove that \mathbb{R}/\mathbb{Q} has the trivial topology.

2. Determine whether the following spaces are homeomorphic. Make sure that you provide a proof:

- (a) The spaces

$$X := \{x \in \mathbb{R} : |x| > 0\} \quad \text{and} \quad Y := \{x \in \mathbb{R} : x > 0\}.$$

- (b) The spaces

$$X := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \quad \text{and} \quad Y := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

- (c) The spaces

$$X := \mathbb{R}^2 \quad \text{and} \quad Y := \mathbb{R}^2 - \{0\}.$$

3. Consider the projection map

$$\pi : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad (x, y) \mapsto x.$$

Prove that this is *not* a closed mapping (taking closed sets to closed sets).

4. Consider the interval $X := [0, 1]$ and the subspace

$$A := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$

of X . Consider the quotient X/A , that is the quotient of X by the relation

$$x \sim y \Leftrightarrow (x \in A \wedge y \in A).$$

Prove that X/A is *not* Hausdorff.

5. Using the Seifert-Van Kampen Theorem, compute the fundamental group of the Klein bottle. Recall that the Klein bottle is obtained from a cylinder by gluing the ends using a reflection.
6. Give a complete proof of Urysohn's Lemma.
7. Let $\rho : \tilde{X} \rightarrow X$ be a covering space mapping a basepoint $\tilde{x}_0 \in \tilde{X}$ to a basepoint $x_0 \in X$. Prove that any path $\gamma : [0, 1] \rightarrow X$ with initial point x_0 has a lift $\tilde{\gamma} : [0, 1] \rightarrow \tilde{X}$ starting at \tilde{x}_0 ,

$$\rho \circ \tilde{\gamma} = \gamma.$$

8. Using Mayer-Vietoris or otherwise, compute the homology of the spheres S^n for $n \geq 0$.