Topology Preliminary Exam
August 2025

Directions

Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any major theorem is used in any argument, give a precise statement of that theorem.

There are eight questions on the exam. You will be graded on your seven best answers.

Problems

1. Consider the nonnegative real line $[0, \infty) \subset \mathbb{R}$ and its quotient by the subspace $[1, \infty)$:

$$x \sim y \Leftrightarrow (x \in [1, \infty) \quad \text{and} \quad y \in [1, \infty)).$$

   Prove that the quotient $\mathbb{R}/[1, \infty)$ is homeomorphic to the interval $[0, 1]$.

2. Determine whether the following spaces are homeomorphic. Make sure that you provide a proof:

   (a) The spaces

$$X := \{z \in \mathbb{C} : |z| \leq 1\} \quad \text{and} \quad Y := \{z \in \mathbb{C} : |z| < 1\}.$$

   (b) The real line $\mathbb{R}$ and the plane $\mathbb{R}^2$.

   (c) The subspaces of integers $\mathbb{Z}$ and of rationals $\mathbb{Q}$ of the real line $\mathbb{R}$.

3. Let X be a Hausdorff topological space:

   (a) Prove that for every point $x \in X$ and compact subset $K \subset X$ which does not contain x, there are disjoint open subsets U and V of X such that

$$x \in U \quad \text{and} \quad K \subset V.$$

(b) Hence show that any compact Hausdorff topological space is regular.

4. Consider the lines in the plane given by

$$L_n := \left\{(x,y) \in \mathbb{R}^2 : x \geq 0 \quad \text{and} \quad y = \frac{x}{n}\right\}, \quad n = 1, 2, \ldots .$$

Set

$$x_0 := (1,0) \quad \text{and} \quad C := \{x_0\} \cup \bigcup_{n=1}^{\infty} L_n.$$

Prove that C is a connected subset of $\mathbb{R}^2$.

5. Give a complete proof of Urysohn's Lemma

6. Let $\rho : \widetilde{X} \to X$ be a covering space mapping a basepoint $\tilde{x}_0 \in \widetilde{X}$ to a basepoint $x_0 \in X$. Prove that any path $\gamma : [0,1] \to X$ with initial point x_0 has a lift $\tilde{\gamma} : [0,1] \to \widetilde{X}$ starting at $\tilde{x}_0$,

$$\rho \circ \tilde{\gamma} = \gamma.$$

7. Recall that the real projective plane $\mathbb{RP}^2$ is the quotient of the sphere S^2 by the action of the cyclic group of order two that identifies antipodal points. Find all the covering spaces of the product $\mathbb{RP}^2 \times \mathbb{RP}^2 \times \mathbb{RP}^2$.

8. Consider the two spheres

$$S_1 := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z-1)^2 = 1\}, \quad \text{and}$$
$$S_2 := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + (z+1)^2 = 1\}$$

and denote their union by $S := S_1 \cup S_2$. Using Mayer-Vietoris or otherwise, compute the singular homology groups of S.