Topology Preliminary Exam
May 2025

Directions

Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any major theorem is used in any argument, give a precise statement of that theorem.

There are eight questions on the exam. You will be graded on your seven best answers.

Problems

1. Consider the real line $\mathbb{R}$ and its quotient by the subspace of rational numbers $\mathbb{Q}$:
$$x \sim y \Leftrightarrow (x \in \mathbb{Q} \quad \text{and} \quad y \in \mathbb{Q}).$$
Prove that the quotient is not Hausdorff.

2. Determine whether the following spaces are homeomorphic. Make sure that you provide a proof:

   (a) The spaces
   $$X := \{z \in \mathbb{C} : 1 \leq |z| \leq 2\} \quad \text{and} \quad Y := \{z \in \mathbb{C} : 1 < |z| < 2\}.$$

   (b) The circle and the sphere
   $$S^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \quad \text{and}$$
   $$S^2 := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

   (c) The spaces
   $$X := \{(x, y) \in \mathbb{R}^2 : |x| < 1\} \quad \text{and} \quad Y := \{(x, y) \in \mathbb{R}^2 : |x| > 1\}.$$

3. Let $x_n, n \in \mathbb{N}$ be a sequence of points in a metric space X that converges to a point $x \in X$. Prove that the set

$$K := \{x\} \cup \{x_n : n \in \mathbb{N}\}$$

is a compact subset of X.

4. Consider the lines in the plane given by

$$L_n := \left\{(x, y) \in \mathbb{R}^2 : x \geq 0 \quad \text{and} \quad y = \frac{x}{n}\right\}, \quad n = 1, 2, \ldots.$$

Set

$$x_0 := (1, 0) \quad \text{and} \quad C := \{x_0\} \cup \bigcup_{n=1}^{\infty} L_n.$$

Prove that C is a connected subset of $\mathbb{R}^2$.

5. Give a complete proof of the Tietze Extension Theorem.

6. Let $\rho : \widetilde{X} \to X$ be a covering space mapping a basepoint $\tilde{x}_0 \in \widetilde{X}$ to a basepoint $x_0 \in X$. Prove that any path $\gamma : [0, 1] \to X$ with initial point x_0 has a lift $\tilde{\gamma} : [0, 1] \to \widetilde{X}$ starting at $\tilde{x}_0$,

$$\rho \circ \tilde{\gamma} = \gamma.$$

7. Consider the unit disc and the unit circle in the complex plane,

$$D := \{z \in \mathbb{C} : |z| \leq 1\} \quad \text{and} \quad S^1 := \{z \in \mathbb{C} : |z| = 1\}.$$

Consider the action of $\mathbb{Z}/n\mathbb{Z}$ on S^1;

$$k \cdot z := e^{\frac{2k\pi i}{n}} z, \quad k \in \mathbb{Z}/n\mathbb{Z}, z \in S^1.$$

Let Y be the quotient of D by the relation

$$z \sim k \cdot z, \quad k \in \mathbb{Z}/n\mathbb{Z}, z \in S^1.$$

Note that only points on the boundary S^1 are related. Using the Seifert-Van Kampen Theorem or otherwise, compute the fundamental group of Y.

8. Consider the two circles

$$C_1 := \{z \in \mathbb{C} : |z| = 1\} \quad \text{and} \quad C_2 := \{z \in \mathbb{C} : |z - 1| = 1\}$$

and denote their union by $C := C_1 \cup C_2$. Using Mayer-Vietoris or otherwise, compute the singular homology groups of C.