TOPOLOGY
PRELIMINARY EXAM
May, 1996

WORK ALL PROBLEMS. CLEARLY INDICATE ANY MAJOR RESULTS USED. ASSUME ALL SPACES ARE T_2.

1. Give an example of each of the following. Clearly indicate why your example has the desired properties.

a.) a metric space with no equivalent complete metric,
b.) a separable space which is not Lindelöf,
c.) a space in which every infinite set has a limit point, but not every sequence has a convergent subsequence.

2. State and prove the Tietze extension theorem.

3. Let d be a metric compatible with the topology on the space X. Show that $d : X \times X \to R$ is a continuous function.

4. Let $X = \prod_{\alpha \in A} X_\alpha$. Prove that X is connected if and only if each X_α is connected.

5. Prove that if X is second countable then every base for the topology of X has a countable subcollection which is a base for that topology.

6. Show that if $f : X \to Y$ is a closed, continuous, surjection with X locally compact and each $f^{-1}(y)$ compact, then Y is locally compact.

7. Let $p : (E, e_0) \to (X, x_0)$ be a covering space map of the path connected space X. Show that if $p^{-1}(x_0)$ has exactly k elements, then $p^{-1}(x)$ has exactly k elements for each $x \in X$.

8. A path connected space X is *1-simple* if and only if every two paths p and q in X with $p(0) = q(0)$ and $p(1) = q(1)$ induce the same isomorphism from $\pi_1(X, p(0))$ to $\pi_1(X, p(1))$, i.e. $[p^r][\alpha][p] = [q^r][\alpha][q]$ for each loop α based at $p(0)$. Show that X is 1-simple if and only if $\pi_1(X)$ is abelian.

9. Suppose that X and Y are topological spaces and that $f : X \to Y$ is a homotopy equivalence with homotopy inverse $g : Y \to X$. Show that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, f(x_0))$. [**WARNING:** While $gf \sim id_X$, the homotopy need not keep x_0 fixed.]