TOPOLOGY PRELIMINARY EXAM
MAY 1997

WORK ALL PROBLEMS. CLEARLY REFERENCE ANY MAJOR THEOREMS OR RESULTS USED.

Assume All Spaces Are T_2.

1) Show that any compact subspace of a Hausdorff space is closed.

2) Let (X, d) be a metric space. Show that $d : X \times X \to \mathbb{R}$ is continuous.

3) Let $X = \prod_{\alpha \in A} X_\alpha$, where each X_α is nonempty. Show that X is connected if and only if each X_α is connected.

4) Give examples of each of the following. Indicate why your examples have the indicated properties.

(a) A metric space having no equivalent complete metric.

(b) A space showing that the product of normal spaces need not be normal.

(c) A space that is completely regular but is not normal.

(d) A space whose quasi-components are not the same as its components.

5) Let $f : X \to Y$ be a continuous surjection and let X be locally connected and compact Hausdorff. Prove that Y is locally connected.

6) State and prove the Baire Category Theorem for complete metric spaces.

7) Show the following without using the Van Kampen Theorem. Let $X = U \cup V$, where U and V are open in X and $U \cap V$ is path connected. Let x_0 be a point of $U \cap V$. If both inclusions:

(i) $i : (U, x_0) \to (X, x_0)$, and

(ii) $j : (V, x_0) \to (X, x_0)$,

induce zero homomorphisms of the fundamental groups, then $\pi_1(X, x_0) = 0$.

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8) Let $p : E \to B$ be a covering map and let $p(e_0) = b_0$. Prove that any path $f : [0,1] \to B$ beginning at b_0 has a unique lifting to a path $\tilde{f}$ in E beginning at e_0. (This result is sometimes called the path lifting lemma.)

9) Prove that the following statements are equivalent. Prove that one of these statements is true.

(a) Any continuous function from the disk to itself has a fixed point.

(b) There is no retract of a disk $D = \{(x,y) \in \mathbb{R}^2 | \sqrt{x^2 + y^2} \leq 1\}$ onto its boundary.