TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
August 2003

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE A PRECISE STATEMENT OF ANY MAJOR THEOREM REFERENCED IN ANY ARGUMENT. GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE.

1.) Let $U = \{U_\alpha | \ \alpha \in A\}$ be an open cover of the compact metric space (X, d). Show that there exists a number $\delta > 0$ such that for every subset H of X with $\text{diam}(H) < \delta$ there exists $\alpha_0 \in A$ such that $H \subset U_{\alpha_0}$.

2.) Let (X, d) be a metric space. Show that the following are equivalent.
a.) X has a countable dense subset.
b.) X has a countable basis for its topology.
c.) Every open cover of X has a countable subcover.

3.) Let $X = \prod_{\alpha \in A} X_\alpha$, where A is an arbitrary indexing set and each X_α is nonempty. Show that X is regular if and only if each X_α is regular.

Give an example to show that the product of normal spaces need not be normal. Clearly indicate why your example has the desired properties.

4.) Show that if $f : X \longrightarrow Y$ is a closed, continuous surjection with X locally compact and each $f^{-1}(y)$ compact, then Y is locally compact.

Show that if the hypothesis that each $f^{-1}(y)$ is compact is omitted then Y need not be locally compact.

5.) Let $X = \prod_{\alpha \in A} X_\alpha$, where A is an arbitrary indexing set and each X_α is nonempty. Prove that X is connected if and only if each X_α is connected.

6.) Let $p : (E, e_0) \to (B, b_0)$ be a covering map of the path connected space B. Show that if $p^{-1}(b_0)$ has exactly k elements, then $p^{-1}(b)$ has exactly k elements for each $b \in B$.

7.) Let $h : S^1 \to S^1$ be a nullhomotopic continuous function from the unit circle S^1 to itself. Show that h has a fixed point and that h maps some point $x \in S^1$ to its antipode $-x$.

8.) Assume that each of X_1, X_2 and $X_1 \cap X_2$ is an arcwise-connected open subset of the space X, where $X = X_1 \cup X_2$ and $x_0 \in X_1 \cap X_2$. Let $i : X_1 \to X$ and $j : X_2 \to X$ be the inclusion mappings of X_1 and X_2, respectively, into X. Show that the images of the induced homomorphisms $i_* : \pi_1(X_1, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(X_2, x_0) \to \pi_1(X, x_0)$ generate $\pi_1(X, x_0)$. (This is a major step in the proof of the Seifert - van Kampen theorem. Do not quote this theorem as part of the above argument.)

Using this result, give a presentation for the fundamental group of the surface represented by the two-holed torus.