TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
August 2004

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE.

1.) Give an example of each of the following. Clearly describe the space and topology for each example and why it has the indicated properties.

a.) A space X with a countable dense subset but no countable basis for its topology.

b.) A space Y in which every infinite set has a limit point but not every sequence has a convergent subsequence.

c.) A noncompact space Z on which every continuous real-valued function is bounded.

2.) Let (X, d_1) be a compact metric space, (Y, d_2) be a metric space and $f : X \to Y$ be a continuous function. Show that if $\epsilon > 0$ such that $\text{diam}(f^{-1}(y)) < \epsilon$ for each $y \in Y$, then there exists $\delta > 0$ such that $\text{diam}(f^{-1}(A)) < \epsilon$ for each subset A of Y with $\text{diam}(A) < \delta$.

3.) Let X be locally compact and $\{H_n\}_{n=1}^{\infty}$ be a collection of closed subsets of X such that each H_n has empty interior in X. Show that $\bigcup_{n=1}^{\infty} H_n \neq X$ (i.e., X is a Baire space).

4.) Let X be a locally connected space and $f : X \twoheadrightarrow Y$ a closed continuous surjection. Show that Y is locally connected.

5.) Let X and Y be topological spaces with Y compact. Let $x_0 \in X$ and let U be an open set in $X \times Y$ such that $(\{x_0\} \times Y) \subset U$. Show that there exists an open set V in X such that $x_0 \in V$ and $(V \times Y) \subset U$.

6.) Let Y, E and B be path connected, locally path connected spaces with $y_0 \in Y$, $e_0 \in E$ and $b_0 \in B$. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$ and let $f : Y \to B$ be a continuous function with $f(y_0) = b_0$. Show that there exists a continuous function $\tilde{f} : Y \to E$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = e_0$ if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$.

7.) Prove that if $f : S^1 \to S^1$ is a continuous function from the unit circle S^1 to itself which is antipode-preserving (i.e., $f(-x) = -f(x)$ for each $x \in S^1$), then f is not homotopic to a constant.

8.) Let $X = U \cup V$, where U and V are open in X, each of U, V and $U \cap V$ is path connected and $x_0 \in U \cap V$. Show that if V is simply connected then there is an isomorphism $k : \pi_1(U, x_0)/N \to \pi_1(X, x_0)$, where N is the smallest normal subgroup of $\pi_1(U, x_0)$ containing the image of the homomorphism $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$. (This is a corollary of the Seifert-van Kampen theorem. Do not simply quote this theorem, but give a direct proof of the above result.)