TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
May 2004

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE.

1.) Let (X, d) be a metric space. Show that the following are equivalent.
a.) X is Lindelöf.
b.) X is separable.
c.) X is second countable.
Give an example of a space which is separable but not Lindelöf.

2.) Let (X, d) be a compact metric space. Show that each uncountable subset of X contains a limit point. (Do not simply show that such a set has a limit point somewhere in X.)

3.) Let $X = \prod_{\alpha \in A} X_\alpha$ have the standard product topology, where A is an arbitrary indexing set and each X_α is nonempty. Let $f : Y \to \prod_{\alpha \in A} X_\alpha$ be given by the equation $f(y) = (f_\alpha(y))_{\alpha \in A}$, where $f_\alpha : Y \to X_\alpha$ for each $\alpha \in A$. Show that f is continuous if and only if each function f_α is continuous.

4.) Let $\mathbb{R}^\omega$ be the product of countably many copies of the real line $\mathbb{R}$ (equivalently, the collection of all sequences of real numbers).
a.) Show that $\mathbb{R}^\omega$ is connected and path connected in the standard product topology.
b.) Show that if $\mathbb{R}^\omega$ has the topology induced by the uniform metric then points $\mathbf{x}$ and $\mathbf{y}$ are in the same component of $\mathbb{R}^\omega$ if and only if the sequence $\mathbf{x} - \mathbf{y} = (x_1 - y_1, x_2 - y_2, \dots)$ is bounded.
c.) Show that if $\mathbb{R}^\omega$ has the box topology then points $\mathbf{x}$ and $\mathbf{y}$ are in the same component of $\mathbb{R}^\omega$ if and only if the sequence $\mathbf{x} - \mathbf{y}$ is eventually zero.

5.) Let $f : X \to Y$ be a closed continuous surjection such that $f^{-1}(\{y\})$ is compact for each $y \in Y$. Show that if Y is compact then X is compact.

6.) Assume that each of X, Y and Z is locally path connected and path connected. Let $p : X \to Y$ and $q : Y \to Z$ be covering maps. Show that if $q^{-1}(z)$ is finite for each $z \in Z$ then $r = q \circ p$ is a covering map.

7.) Let $h : S^1 \to S^1$ be a nullhomotopic continuous function from the unit circle S^1 to itself. Show that h has a fixed point and that h maps some point $x \in S^1$ to its antipode $-x$.

8.) Assume that each of X_1, X_2 and $X_1 \cap X_2$ is an arcwise-connected open subset of the space X, where $X = X_1 \cup X_2$ and $x_0 \in X_1 \cap X_2$. Show that if each of X_1 and X_2 is simply-connected then X is simply-connected. (This is a special case of the Seifert-van Kampen Theorem. Do not quote this theorem for the above, but rather give a direct argument not using this stronger theorem.)