TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
May 2005

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE.

1.) Give an example of each of the following. Clearly describe the space and topology for each example and why it has the indicated properties.

a. A space X without a countable dense subset but for which every open cover has a countable subcover.

b. Normal spaces Y and Z whose product $Y \times Z$ is not normal.

c. A noncompact space W on which every continuous real-valued function is bounded.

2.) Show that every regular, Lindelöf space is normal.

3.) Let X be a metric space. Show that the following are equivalent.

a. X has a countable dense subset.

b. X has a countable basis for its topology.

c. Every open cover of X has a countable subcover.

4.) a. If X is normal and y is a point of $\beta X - X$ (βX is the Stone-Čech compactification of X), show that while y is a limit point of X there is no sequence of points in X which converges to y.

b. Show that if X is completely regular and noncompact then its Stone-Čech compactification is not metrizable.

5.) Let X be a topological space and (Y, d) be a metric space. Let $\mathcal{F}$ be a subset of Y^X which is equicontinuous. (Y^X is the collection of all functions, not necessarily continuous, from X to Y.) Show that if $\mathcal{G}$ is the closure of $\mathcal{F}$ in Y^X in the point-open topology, then the family of functions $\mathcal{G}$ is equicontinuous.

6.) Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Show that if E is path connected then the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. Show that if E is simply connected then this correspondence is a bijection.

7.) Prove that if $f : S^2 \to \mathbb{R}^2$ is a continuous function from the 2-dimensional sphere S^2 to the plane, then there is a point $x \in S^2$ such that $f(x) = f(-x)$ (where $-x$ denotes the antipode of x in S^2). (The result that there is no antipode preserving continuous function $g : S^2 \to S^1$ from the 2-dimensional sphere to the circle is equivalent to the above result and is not to be used by itself to prove the above result.)

8.) Give a presentation for the fundamental group $\pi_1(X)$ for each of the following spaces. Justify your results.

a. The two-holed torus.

b. The Klein bottle.

c. The 1-skeleton of a 3-simplex.