# TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

**August 2007**

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF ($T_2$). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. (a) Let $X$ be a topological space. Show that $X$ is regular if and only if given a point $x$ in $X$ and a neighborhood $U$ of $x$, there is a neighborhood $V$ of $x$ such that $\overline{V} \subset U$.
   (b) Let $X = \prod_{\alpha \in A} X_\alpha$, where each $X_\alpha$ is a regular topological space. Show that $X$ is regular.

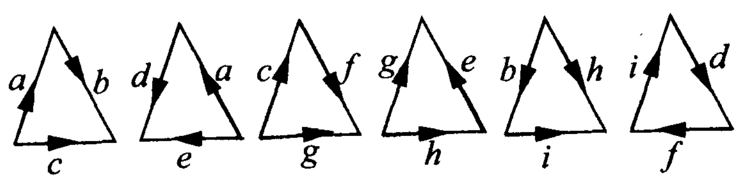
2. Show that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected and locally path connected.

3. Let $X$ have a countable basis, and let $A$ be an uncountable subset of $X$. Show that uncountably many points of $A$ are limit points of $A$.

4. State and prove the Lebesgue number lemma.

5. Write the torus as a semisimplicial (i.e. $\Delta$)-complex, then compute its homology with coefficients in $\mathbb{Z}$.

6. Identify, according to the classification theorem of surfaces, the surface obtained by gluing the following triangles along edges as specified:




7. Find, with justification, a presentation of the fundamental group of a genus 2 surface.

8. Consider the path connected and locally path connected spaces $E, B, Y$. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Also, let $f : Y \to B$ be a continuous map, with $f(y_0) = b_0$. Show that the map $f$ can be lifted to a map $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ if and only if

$$f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0)).$$

   Show furthermore that if such a lifting exists, it is unique.