TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

May 2007

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. Let (X, d) be a metric space. Show that the following statements are equivalent:
   a) Every open cover of X has a finite subcover.
   b) Every sequence in X has a convergent subsequence.
   c) Every continuous function $f : X \to \mathbf{R}$ is bounded.

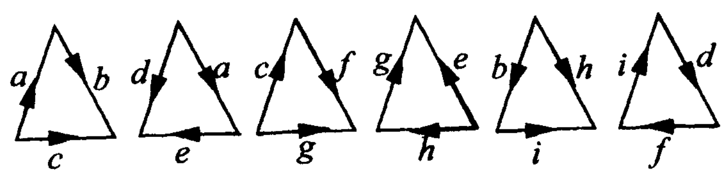
2. State and prove the Urysohn lemma.

3. Consider the product space $X \times Y$, where Y is compact. If U is an open set in $X \times Y$ containing the set $\{x_0\} \times Y$, show that there exists a "tube" $V \times Y$ such that $\{x_0\} \times Y \subset V \times Y \subset U$, where V is an open set in X containing x_0.

4. Let A and X be two connected topological spaces such that $A \subset X$. Show that if H and K form a separation of $X \backslash A$, then each of $A \cup H$ and $A \cup K$ is connected.

5. Write the Klein bottle as a semisimplicial (i.e. Δ)-complex, then compute its homology with integer coefficients.

6. Identify the surface obtained by gluing the following triangles along edges as specified, according to the classification theorem of surfaces:



7. Find, with justification, the fundamental group of the (a) projective plane and (b) the torus.

8. Let E and B be path connected spaces with $e_0 \in E$ and $b_0 \in B$. Let $p : E \to B$ be a covering map with $p(e_0) = b_0$ and let $f : [0, 1] \to B$ be a continuous function with $f(0) = b_0$. Show that there exists a unique continuous function $\tilde{f} : [0, 1] \to E$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(0) = e_0$.