TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
AUGUST 2009

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1.) Let $f : X \twoheadrightarrow Y$ be a closed continuous surjection. Show that if X is locally connected then Y is locally connected.

2.) Let X be second countable, i.e. let there be a countable base of open sets for the topology on X. Show that X is Lindelöf, i.e. show that every open cover of X has a countable subcover.

Give an example of a Lindelöf space which is not second countable.

3.) Let $X = \prod_{\alpha \in A} X_\alpha$ have the standard product topology, where A is an arbitrary nonempty index set and each X_α is nonempty. Let $f : Y \to \prod_{\alpha \in A} X_\alpha$ be given by $f(y) = (f_\alpha(y))_{\alpha \in A}$, where $f_\alpha : Y \to X_\alpha$ for each $\alpha \in A$. Show that f is continuous if and only if each f_α is continuous.

4.) Let X and Y be topological spaces with Y compact. Let $x_0 \in X$ and let U be an open set in $X \times Y$ such that $(\{x_0\} \times Y) \subset U$. Show that there exists an open set V in X such that $x_0 \in V$ and $(V \times Y) \subset U$.

Give an example to show that no such open set V need exist if Y is not compact.

5.) Let X be a topological space and let $\{X_n\}_{n=1}^{\infty}$ be a collection of subspaces of X such that each X_n is nonempty, compact and connected and $X_{n+1} \subset X_n$ for each n. Show that $\bigcap_{n=1}^{\infty} X_n$ is nonempty, compact and connected.

Give an example to show that if the X_n's are not compact then $\bigcap_{n=1}^{\infty} X_n$ need not be connected.

6.) Prove that if $f : S^1 \twoheadrightarrow S^1$ is a continuous function from the unit circle S^1 to itself which is antipode-preserving (i.e., $f(-x) = -f(x)$ for each $x \in S^1$), then f is not homotopic to a constant.

7.) Let $p : E \twoheadrightarrow B$ be a covering map with $p(e_0) = b_0$. Show that if E is path connected then the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. Show that if E is simply connected then this correspondence is a bijection.

8.) Let $f : X \to Y$ be continuous with $f(x_0) = y_0$. Show that if f is a homotopy equivalence then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism of fundamental groups. (Note that the homotopies showing a homotopy equivalence do not necessarily preserve base points.)