TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
MAY 2009

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1.) Let $f : X \twoheadrightarrow Y$ be a continuous surjection.
  a) Show that if X is separable then Y is also separable.
  b) Show that if X is Lindelöf then Y is also Lindelöf.

2.) Let $X = \prod_{\alpha \in A} X_\alpha$ where each X_α is nonempty and A is an arbitrary nonempty index set. Prove that X is connected if and only if each X_α is connected.

3.) Let (X, d) be a metric space. Show that the following statements are equivalent.
  a) X contains a countable dense subset.
  b) X has a countable base for its topology.
  c) Every open cover of X has a countable subcover.

4.) Prove the Baire Category Theorem for compact Hausdorff spaces, i.e. show that if $\{B_n\}_{n=1}^{\infty}$ is a collection of dense open subsets of the compact Hausdorff space X then $\bigcap_{n=1}^{\infty} B_n$ is dense in X.

5.) Give an example of each of the following. Clearly describe the space and topology for each example and why it has the indicated properties.
  a) A regular space X which is not normal.
  b) A space Y which is Lindelöf but which is not separable.
  c) A space Z in which every infinite set has a limit point but not every sequence has a convergent subsequence.

6.) Let $p : E \twoheadrightarrow B$ be a covering map, with $p(e_0) = b_0$. Let f and g be paths in B from b_0 to b_1, with $\tilde{f}$ and $\tilde{g}$ their respective liftings to paths in E beginning at e_0. Show that if f and g are path homotopic in B then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic in E.

7.) Let $X = U \cup V$, where each of U and V is open in X, each of U, V and $U \cap V$ is path connected, $x_0 \in U \cap V$ and each of U and V is simply connected. Show that X is simply connected. (This is a corollary of the Seifert-van Kampen theorem. Do not simply quote this theorem, but give a direct proof of the above result.)

8.) Prove each of the following statements.
  a) There is no retract of the unit disk B^2 to the unit circle S^1.
  b) Every continuous function from the unit disk B^2 to itself has a fixed point.