TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION AUGUST 2010

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1.) Give an example of each of the following. Clearly describe the space and its topology and indicate why it does or does not have the indicated properties.

a) a regular space which is not normal.

b) a Lindelöf space which is neither separable nor second countable.

c) a noncompact space in which every sequence has a convergent subsequence.

2.) Prove the Tietze Extension Theorem: If A is a closed subset of the normal space X and $f : A \to [0,1]$ is continuous, then there exists a continuous function $\tilde{f} : X \to [0,1]$ such that $\tilde{f}(a) = f(a)$ for each $a \in A$.

3.) Let X_1 and X_2 be nonempty topological spaces and let $\pi_1 : X_1 \times X_2 \to X_1$ be projection onto the first coordinate. Show that if X_2 is compact then π_1 is a closed continuous surjection.

Give an example to show that if X_2 is not compact then π_1 need not be closed.

4.) Show that each closed subset of a compact space is compact.

Show that each compact subset of a Hausdorff space is closed.

5.) Let $X = \prod_{\alpha \in A} X_\alpha$ have the standard product topology, where A is an arbitrary nonempty index set and each X_α is nonempty. Show that X is connected if and only if each X_α is connected.

6.) Let $X = U \cup V$, where each of U and V is a path connected open set in X. Let $U \cap V$ be path connected with $x_0 \in U \cap V$. Let i and j be the inclusion mappings of U and V, respectively, into X. Show that the images of the induced homomorphisms $i_* : \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$ generate $\pi_1(X, x_0)$. (This is a consequence of the Seifert-van Kampen theorem. However, the above result is a critical step in the proof of the Seifert-van Kampen theorem. Do not simply quote the Seifert-van Kampen theorem, but prove the above result directly.)

7.) Assume that each of X, Y and Z is locally path connected and path connected. Let $p : X \to Y$ and $q : Y \to Z$ be covering maps. Show that if $q^{-1}(z)$ is finite for each $z \in Z$ then $r = q \circ p$ is a covering map.

8.) Prove that if $f : S^2 \to \mathbb{R}^2$ is a continuous function from the 2-dimensional sphere S^2 to the plane, then there is a point $x \in S^2$ such that $f(x) = f(-x)$, where $-x$ denotes the antipode of x in S^2. (The result that there is no antipode preserving continuous function $g : S^2 \to S^1$ from the 2-dimensional sphere to the circle is equivalent to the above result and is not to be used by itself to prove the above result.)