TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

August 2011

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. a) Give an example of a continuous function $f : X \to Y$ for which there is an open set $U \subset X$ such that $f(U)$ is not open.
   b) Give an example of a continuous function $f : X \to Y$ such that there is a closed set $A \subset X$ for which $f(A)$ is not closed.
   c) Give and example of a continuous function $f : X \to Y$ that is not a homeomorphism with the property that for every open set $U \subset X$, $f(U)$ is open and for every closed set $A \subset X$, $f(A)$ is closed.

2. Prove that the product of finitely many compact spaces is compact.

3. a) Show that a topological space X is regular if and only if given a point $x \in X$ and an open neighborhood U of x, there is an open neighborhood V of x such that $\overline{V} \subset U$.
   b) Show that a topological space X is normal if and only if given a closed set $C \subset X$ and an open set U containing C, there is an open set V containing C such that $\overline{V} \subset U$.

4. a) Show that $\mathbb{R}$ and $\mathbb{R}^2$ are not homeomorphic.
   b) Show that $\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

5. State and prove the path lifting lemma.

6. Use the Seifert-van Kampen theorem to compute the fundamental group of the Klein bottle.

7. Compute the homology groups with integer coefficients of the projective plane $\mathbb{R}P^2$.

8. a) Compute the homology groups with real coefficients of the wedge of three circles.
   b) Compute the Euler characteristic of $S^2 \bigvee S^2$ (the wedge of two 2-dimensional spheres, obtained by identifying a point on the first sphere with a point on the second sphere).