TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

May 2011

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. a) Give an example of a space that is connected but not path connected.
   b) Give an example of a space that is path connected but not locally path connected.
2. Consider the product space $X \times Y$, where Y is compact. Show that if $x_0 \in X$, and N is an open set of $X \times Y$ containing $\{x_0\} \times Y$, then N contains a set of the form $W \times Y$ with W an open set in X containing x_0.
3. State and prove Urysohn's lemma.
4. Let X be a path connected space and $x_0, x_1 \in X$. Show that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.
5. a) State and prove the general lifting lemma.
   b) Use the general lifting lemma to show that every continuous map from the 2-dimensional sphere to the 2-dimensional torus is null-homotopic.
6. Use the Seifert-van Kampen theorem to compute the fundamental group of a sphere with two handles (a genus 2 surface).
7. Compute the homology groups with integer coefficients of the torus.
8. a) Compute the homology groups with real coefficients of the Klein bottle.
   b) What is the Euler characteristic of the Klein bottle?