TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

August 2012

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. a) Give an example of a basis for the standard topology on $\mathbb{R}$.
   b) Give an example of a basis for the discrete topology on $\mathbb{R}$.
   c) Let $f : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be given by $f(x) = (x, 2x)$. If $\mathbb{R}$ is endowed with the discrete topology and $\mathbb{R} \times \mathbb{R}$ is endowed with the standard topology, is f continuous? Explain your answer.

2. a) Prove that the product of two compact spaces is compact.
   b) Show that a subset of a compact space is compact if and only if it is closed.

3. a) Show that a topological space X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.
   b) Show that a topological space X is normal if and only if given a closed set $C \subset X$ and an open set U containing C, there is an open set V containing C such that $\overline{V} \subset U$.

4. State and prove Urysohn's lemma.

5. Show that if the path connected topological spaces X and Y are homotopically equivalent, then their fundamental groups are isomorphic.

6. Use the Seifert-van Kampen theorem to compute the fundamental group of the torus.

7. Compute the homology groups with integer coefficients of the Klein bottle.

8. Use the fundamental group to prove the Gauss-d'Alembert fundamental theorem of algebra, which states that every non-constant polynomial with complex coefficients has at least one complex zero.