TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

May 2012

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. a) Let $f : X \to Y$ be a continuous bijective function. Show that if X is compact, then f is a homeomorphism.
   b) Show that if $f : X \to \mathbb{R}$ is continuous and X is compact, then f has a maximum and a minimum.

2. a) Show that the union of a family of connected spaces that have a common point is connected.
   b) Show that the product of two connected spaces is connected.
   c) Show that the image of a connected set through a continuous map is connected.

3. State and prove the Tietze extension theorem.

4. Prove that no two of the following spaces are homeomorphic:
   a) The unit interval $(0, 1)$.
   b) The unit disk $D = \{(x, y) \,|\, x^2 + y^2 < 1\}$.
   c) The annulus $A = \{(x, y) \,|\, 1 \leq x^2 + y^2 \leq 4\}$.

5. Let X be a path connected space and $x_0, x_1 \in X$. Show that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

6. State and prove the homotopy lifting lemma.

7. a) Compute the fundamental group of the projective plane.
   b) Compute the homology groups with integer coefficients of the projective plane.

8. What are the universal covering spaces of the following topological spaces:
   a) $\mathbb{R} \times \mathbb{R}$,
   b) $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$,
   c) $S^1 \times \mathbb{R}$,
   d) $S^1 \times S^1$.

   Explain your answers.