TOPOLOGY PRELIMINARY EXAM
MAY 2013

Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any major theorem is used in any argument, give a precise statement of that theorem. Do all 8 problems.

(1) (a) (i) Define what it means for a map $f : X \to Y$ between two topological spaces X and Y to be an open map.
    (ii) Suppose that $\mathcal{B}$ is a base for a topological space. Show that f is an open map if and only if $f(B)$ is open for every $B \in \mathcal{B}$.
  (b) Give an example, together with a brief justification (not a proof), of a space X such that:
    (i) X is path connected but not locally path connected.
    (ii) X is paracompact but not compact.
    (iii) X is compact but not Hausdorff.

(2) Prove that a space X is normal if and only if for every open subset U of X and every closed subset E of X such that $E \subset U$, there is an open subset V of X such that

$$E \subset V \quad \text{and} \quad \overline{V} \subset U.$$

(3) Let X be a topological space. Show that $H_0(X)$, the singular homology group in degree zero, is isomorphic to the free abelian group with basis the set P of path connected components of X;

$$H_0(X) \cong \mathbb{Z}P.$$

(4) Show that every paracompact Hausdorff topological space is normal.

(5) State and prove the Tietze extension theorem.

(6) Consider the covering space

$$\rho : \widetilde{X} \to X$$

where $\widetilde{X} := \mathbb{C}$, $X := \mathbb{C} - \{0\}$ and $\rho(z) := \exp(z)$.

Explain, *using arguments involving fundamental groups*, why:

  (a) The map

$$\mathrm{id} : X \to X$$

  does not lift to a continuous map to the cover $\widetilde{X}$.

  (b) The inclusion

$$i : \mathbb{C} - \{[0, \infty)\} \to X$$

  lifts to a continuous map to the cover $\widetilde{X}$.

(7) Compute, carefully and with all necessary details, the fundamental group of the real projective plane $\mathbb{R}P^2$.

(8) Suppose that

$$\rho : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$$

is a covering space. Show that any path $\gamma : [0, 1] \to X$ with initial point x_0 has a unique lift $\tilde{\gamma} : [0, 1] \to \widetilde{X}$ with initial point $\tilde{x}_0$.