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TOPOLOGY PRELIMINARY EXAM
AUGUST 2014

Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any theorem is used in any argument, give a precise statement of that theorem. Do all eight problems. Each problem carries equal weight.

(1) Prove that the continuous image of a compact set is compact.

(2) Prove that path connected spaces are connected.

(3) Compute, carefully and with all necessary details, the fundamental group of the space W formed by gluing two circles together at a point;

$$W := \left\{ \mathbf{x} \in \mathbb{R}^2 : \left\| \mathbf{x} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = 1 \right\} \cup \left\{ \mathbf{x} \in \mathbb{R}^2 : \left\| \mathbf{x} - \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\| = 1 \right\}.$$

(4) In each case, say whether the following two spaces are homeomorphic. Include a brief justification for each answer:

(a) The circle and the closed interval $[0,1]$.

(b) The open interval $(0,1)$ and the closed interval $[0,1]$.

(c) The open intervals $(0,1)$ and $(0,\infty)$.

[**Note:** Your justification need not rise to the level of a formal proof, but it should be more than just heuristic.]

(5) Compute, carefully and with all necessary details, the singular homology of a point.

[**Note:** You must give a direct argument involving the singular homology groups; do not treat the point as a simplicial or cell-complex.]

(6) For each of the following, give an example, together with a brief justification, of a continuous function $f : X \to Y$ between topological spaces such that:

(a) f is bijective but not a homeomorphism,

(b) f is a closed mapping but not an open mapping.

(c) f is an open mapping but not a closed mapping.

(7) State and prove the Tietze extension theorem.

(8) Suppose that

$$\rho : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$$

is a covering space. Show that any path $\gamma : I \to X$ with initial point x_0 has a unique lift $\tilde{\gamma} : I \to \widetilde{X}$ with initial point $\tilde{x}_0$.