# TOPOLOGY PRELIMINARY EXAM
## MAY 2014

**Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any major theorem is used in any argument, give a precise statement of that theorem. Do all eight problems.**

(1) Prove that every compact subspace of a Hausdorff topological space is closed.

(2) Compute, carefully and with all necessary details, the fundamental group of the Klein bottle.

(3) Prove that if $f : X \to Y$ is a continuous bijection between a compact space $X$ and a Hausdorff space $Y$, then $f$ must be a homeomorphism.

(4) Prove that the one-point compactification of the open interval $(0, 1)$ is a circle. Be sure to include an explanation of why the topologies coincide.

(5) Let $X$ be a set.

  (a) Give an example of a topology on $X$ for which the only continuous functions
$$f : X \to \mathbb{R}$$
  are the constant functions.

  (b) Give a careful proof that the only continuous functions in your chosen topology are constant.

(6) Give a statement and proof of Urysohn's Lemma.

(7) Consider the space $\mathbb{R}^2$ with the following equivalence relation;
$$\mathbf{x} \approx \mathbf{y} \Leftrightarrow [\exists \lambda \in (0, \infty) : \mathbf{x} = \lambda \mathbf{y}].$$
Let $X := \mathbb{R}^2/\approx$ be the quotient space (endowed with the quotient topology):

  (a) Prove that any closed subset of $X$ must contain (the equivalence class of) the origin.

  (b) Using part (a) or otherwise, show that:

    (i) $X$ is not Hausdorff.

    (ii) $X$ is normal.

(8) Suppose that
$$\rho : (\widetilde{X}, \tilde{x}_0) \to (X, x_0)$$
is a covering space. Show that any path $\gamma : I \to X$ with initial point $x_0$ has a unique lift $\tilde{\gamma} : I \to \widetilde{X}$ with initial point $\tilde{x}_0$.