

**TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION**  
**AUGUST 2015**

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF ( $T_2$ ). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1.) Let  $X$  be a topological space, let  $\{X_n\}_{n=1}^{\infty}$  be a collection of subspaces of  $X$  such that each  $X_n$  is nonempty, compact and connected and let  $X_{n+1} \subseteq X_n$  for each  $n$ . Show that  $\bigcap_{n=1}^{\infty} X_n$  is nonempty, compact and connected.

2.) Let  $f : X \rightarrow Y$  be a closed continuous surjection such that  $f^{-1}(y)$  is compact for each  $y \in Y$ . Show that, if  $Y$  is compact, then  $X$  is compact.

3.) Let  $X = \prod_{\alpha \in A} X_{\alpha}$ , where each  $X_{\alpha}$  is a regular space. Show that  $X$  is regular.

4.) State and prove the Baire Category theorem for locally compact spaces.

5.) Give an example of each of the following. Clearly describe the set and its topology in each example and clearly indicate why it does or does not have the indicated properties.

a) a noncompact space in which every infinite set has a limit point

b) a space which is Lindelöf (every open cover has a countable subcover) but which is not second countable (does not have a countable basis for its topology)

c) a space which is regular but not normal

6.) Let  $p : E \rightarrow B$  be a covering map, with  $p(e_0) = b_0$ . Show that, if  $E$  is path connected, then the lifting correspondence  $\phi : \pi_1(B, b_0) \rightarrow p^{-1}(b_0)$  is surjective. Show that, if  $E$  is simply connected, then this correspondence is a bijection.

7.) Let  $h : S^1 \rightarrow S^1$  be a continuous function from the unit circle  $S^1$  to itself which is not homotopic to the identity. Show that  $h$  has a fixed point (a point  $x$  such that  $h(x) = x$ ) and that  $h$  maps some point  $y \in S^1$  to its antipode  $-y$ .

8.) Assume that each of  $U$ ,  $V$  and  $U \cap V$  is an arcwise-connected open subset of the space  $X$ , where  $X = U \cup V$  and  $x_0 \in U \cap V$ . Let  $i : U \rightarrow X$  and  $j : V \rightarrow X$  be the inclusion maps of  $U$  and  $V$ , respectively, into  $X$ . Show that the images of the induced homomorphisms  $i_* : \pi_1(U, x_0) \rightarrow \pi_1(X, x_0)$  and  $j_* : \pi_1(V, x_0) \rightarrow \pi_1(X, x_0)$  generate  $\pi_1(X, x_0)$ . (This is a major step in the proof of the Seifert-van Kampen Theorem. Do not simply quote this theorem for the above, but give a direct argument not using this stronger theorem.)