TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
MAY 2015

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1.) Let $X = \prod_{\alpha \in A} X_\alpha$ have the standard product topology, where A is an arbitrary nonempty index set and each X_α is nonempty. Show that X is connected if and only if each X_α is connected.

2.) Let (X, d_1) and (Y, d_2) be compact metric spaces and let $f : X \to Y$ be a continuous function. Show that, if $\epsilon > 0$ such that $\operatorname{diam}(f^{-1}(y)) < \epsilon$ for each $y \in Y$, then there exists $\delta > 0$ such that $\operatorname{diam}(f^{-1}(A)) < \epsilon$ for each subset A of Y with $\operatorname{diam}(A) < \delta$.

3.) Let X be a second countable space, i.e. let X have a countable basis for its topology. Prove that **every** basis for the topology of X has a countable subcollection which is also a basis for the topology.

4.) State and prove the Baire Category theorem for complete metric spaces.

5.) Give an example of each of the following. Clearly describe the set and its topology in each example and clearly indicate why it does or does not have the indicated properties.

a) a space in which every infinite set has a limit point but not every sequence has a convergent subsequence

b) a space which is separable (has a countable dense subset) but which is not second countable (does not have a countable basis for its topology)

c) normal spaces X and Y such that $X \times Y$ is not normal

6.) Let $p : E \to B$ be a covering map, with $p(e_0) = b_0$. Let f and g be paths in B from b_0 to b_1, with $\tilde{f}$ and $\tilde{g}$ their respective liftings to paths in E beginning at e_0. Show that if f and g are path homotopic in B then $\tilde{f}$ and $\tilde{g}$ end at the same point of E and are path homotopic in E.

7.) Prove the equivalence of the following two statements. Prove one of the statements.

a.) There is no retract of a disk to its boundary.

b.) Every continuous function from a disk to itself has a fixed point.

8.) Assume that each of U, V and $U \cap V$ is an arcwise-connected open subset of the space X, where $X = U \cup V$ and $x_0 \in U \cap V$. Show that, if each of X_1 and X_2 is simply-connected, then X is simply-connected. (This is a special case of the Seifert-van Kampen Theorem. Do not simply quote this theorem for the above, but give a direct argument not using this stronger theorem.)