TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
AUGUST 2016

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1.) Let (X, d_1) be a compact metric space, (Y, d_2) be a metric space and $f : X \to Y$ be a continuous function. Show that, if $\epsilon > 0$ such that $\text{diam}(f^{-1}(y)) < \epsilon$ for each $y \in Y$, then there exists $\delta > 0$ such that $\text{diam}(f^{-1}(A)) < \epsilon$ for each subset A of Y with $\text{diam}(A) < \delta$.

2.) Let X_1 and X_2 be nonempty topological spaces and let $\pi_1 : X_1 \times X_2 \to X_1$ be projection onto the first coordinate. Show that, if X_2 is compact, then π_1 is a closed continuous surjection.

(Note that if X_2 is not compact then π_1 need not be closed. Clearly indicate where you use the assumption that X_2 is compact.)

3.) Let X be a locally connected space and $f : X \to Y$ a closed continuous surjection. Show that Y is locally connected.

(The continuous image of a locally connected space is not necessarily locally connected, so clearly indicate where you use that the function f is closed.)

4.) A space X is *second countable* if there exists a countable collection of open sets which form a basis for the topology of X. Show that, if X is second countable, then **every** collection of open sets forming a basis for the topology of X has a countable subcollection which forms a basis for the topology of X.

5.) Give an example of each of the following. Clearly describe the set and its topology in each example and clearly indicate why it does or does not have the indicated properties.

a) a noncompact space in which every infinite set has a limit point,

b) a space which is Lindelöf (every open cover has a countable subcover), but which is not separable (there does not exist a countable dense subset),

c) a regular space which is not normal.

6.) Let $p : E \to B$ be a covering map with $p(e_0) = b_0$. Show that, if E is path connected, then the lifting correspondence $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is surjective. Show that, if E is simply connected, then this correspondence is a bijection.

7.) Prove that, if $f : S^2 \to \mathbb{R}^2$ is a continuous function from the 2-dimensional sphere S^2 to the plane $\mathbb{R}^2$, then there is a point $x \in S^2$ such that $f(x) = f(-x)$ (where $-x$ denotes the antipode of x in S^2). (The result that there is no antipode preserving continuous function $g : S^2 \to S^1$ from the 2-dimensional sphere to the unit circle is equivalent to the above result and is not to be used by itself to prove the above result.)

8.) Assume that each of U, V and $U \cap V$ is an arcwise-connected open subset of the space X, where $X = U \cup V$ and $x_0 \in U \cap V$. Show that, if V is simply connected, then there is an isomorphism $k : \pi_1(U, x_o)/N \to \pi_1(X, x_0)$, where N is the smallest normal subgroup of $\pi_1(U, x_0)$containing the image of the inclusion homomorphism $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$.

(This is a corollary of the Seifert-van Kampen Theorem. Do not simply quote this theorem, but give a direct proof of the above result.)