TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

May 2017

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. a) Give an example of a space that is connected but not path connected.
   b) Show that $\mathbb{R}^2 \setminus \mathbb{Q}^2$ is path connected.

2. Consider the product space $X \times Y$, where Y is compact. Show that if $x_0 \in X$ and N is an open set of $X \times Y$ containing $\{x_0\} \times Y$, then N contains a set of the form $W \times Y$ with W an open set in X containing x_0.

3. Explain why no two of the following spaces are homeomorphic to each other (each has the standard subspace topology):
   (a) the interval $[0, 1]$;
   (b) the interval $(0, 1]$;
   (c) the interval $(0, 1)$;
   (d) the circle S^1.

4. Use the fundamental group to prove the Gauss-d'Alembert fundamental theorem of algebra, which states that every non-constant polynomial with complex coefficients has at least one complex zero.

5. State and prove the path lifting lemma.

6. Use the Seifert-van Kampen theorem to compute the fundamental group of the Klein bottle (which is obtained by gluing two Möbius bands along their boundaries).

7. Compute the homology groups with integer coefficients of the projective plane.

8. a) Compute the homology groups with real coefficients of the torus $S^1 \times S^1$.
   b) Compute the Euler characteristic of the torus $S^1 \times S^1$.