TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

August 2017

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. a) Prove that the image through a continuous map of a path connected space is path connected.
   b) Prove that the image through a continuous map of a compact set is compact.

2. Prove the Lebesgue number theorem.

3. a) Show that a topological space X is regular if and only if given a point $x \in X$ and a neighborhood U of x, there is a neighborhood V of x such that $\overline{V} \subset U$.
   b) Show that a topological space X is normal if and only if given a closed set $C \subset X$ and an open set U containing C, there is an open set V containing C such that $\overline{V} \subset U$.

4. State and prove Urysohn's lemma.

5. Let x_0, x_1 be two points in the path connected space X. Prove that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

6. Use the Seifert-van Kampen theorem to compute the fundamental group of the genus 2 surface.

7. Compute the homology groups with integer coefficients of the wedge of 3 circles.

8. Show that the circle S^1, the 2-dimensional sphere S^2, and the 3-dimensional sphere S^3 are not homeomorphic to each other.