TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION

May 2018

WORK ALL PROBLEMS. ASSUME THAT ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. a) Give an example of a space that is connected but not locally connected.
   b) Give an example of a space that is locally connected but not connected.

2. a) Let X, Y be topological spaces such that X is compact and Y is Hausdorff, and let $f : X \to Y$ be continuous. Prove that $f(X)$ is compact.
   b) Let X be a metric space that is sequentially compact. Is it true that X is compact? Prove your answer.

3. Explain why no two of the following spaces are homeomorphic to each other (the first two have the standard subspace topology):
   (a) the circle S^1;
   (b) the 2-dimensional sphere S^2;
   (c) the plane $\mathbb{R}^2$.

4. State and prove Urysohn's Lemma.

5. Let S^1 denote the unit circle, and let $\bar{B}^2$ denote the closed unit disk in the complex plane.

   (a) Let $f : S^1 \to S^1$ be a continuous map. Prove that the following are equivalent:
   (1) f is null homotopic;
   (2) there is a continuous map $F : \bar{B}^2 \to S^1$ such that $F|S^1 = f$;
   (3) $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$ is the zero map.

   (b) Given a non-vanishing continuous vector field on $\bar{B}^2$, prove that there is a point of S^1 where the vector field points directly inwards and a point of S^1 where it points directly outwards.

6. Use the Seifert-van Kampen theorem to compute the fundamental group of the orientable genus 2 surface.

7. Compute the homology groups with integer coefficients of the torus.

8. Compute the homology groups with real coefficients of the 3-dimensional sphere S^3. What is the Euler characteristic of the 3-dimensional sphere?