Topology Doctoral Preliminary Examination

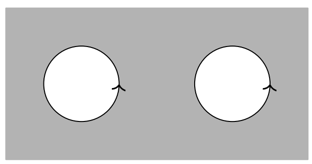
August 2019

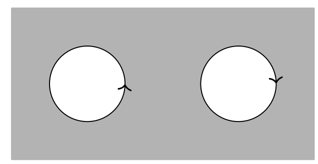
Give complete proofs and computations for all your answers and examples.

1. Three distinct points are identified on a nonorientable surface with 2 crosscaps. Compute the cohomology ring of the resulting space with coefficients in a commutative ring A.

- 1/4 points: cohomology groups computed correctly.
- 3/4 points: cup products in positive degrees computed correctly.

2. Consider two disjoint closed disks in the real projective plane. The interiors of both disks are removed and the newly created boundary circles are identified with each other using two different choices of orientations, as indicated below.





For each of the two possible choices, compute the Poincaré duality isomorphism of the resulting manifold with coefficients in a commutative ring A and classify the resulting surfaces according to the classification of surfaces.

- 1/4 points: homology and cohomology groups computed correctly, surfaces classified correctly.
- 3/4 points: Poincaré duality isomorphisms computed correctly.

3. Classify the following spaces into homeomorphism and homotopy equivalence classes (for all $n \geq 2$): $S^n \times S^n$, $S^n \vee S^n \vee S^{2n}$. (Recall that the wedge $\vee$ of pointed spaces is their disjoint union with all basepoints identified.)

4. Suppose A is a finitely generated abelian group and $n \geq 2$ is an integer. Prove or disprove: there is a simply connected space S (i.e., a connected space with a trivial fundamental group) such that $\mathsf{H}_n(S, \mathbf{Z}) \cong A$ and $\mathsf{H}_i(S, \mathbf{Z}) \cong 0$ for all $i \neq n$, $i > 0$. (You may cite without proof the fact that any such A is a direct sum of finitely many groups isomorphic to $\mathbf{Z}$ or $\mathbf{Z}/m\mathbf{Z}$ for some $m > 1$.)

5. An embedded circle that bounds a regularly embedded disk is removed from the lens space $L(n, p)$. Compute the fundamental group of the resulting space. For each $n \geq 1$ determine whether this space admits a connected Galois covering of degree n.

- 1/2 point: the fundamental group computed correctly.
- 1/2 point: the (non)existence of Galois coverings is established correctly.

6. Give an example of a space and a connected covering of this space that is not a Galois covering. Give an example of a space and a connected covering of this space such that the cardinality of its deck transformation group is not equal to the degree of the covering.

- 1/2 point for each part.