TOPOLOGY PRELIMINARY EXAM
AUGUST 2021

INSTRUCTOR: HAMILTON

Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any major theorem is used in any argument, give a precise statement of that theorem.

There are eight questions on this exam. You will be graded on your seven best answers.

(1) Consider the quotient $X := \mathbb{R}/(0,1)$ of $\mathbb{R}$ by the open unit interval $(0,1)$. Prove that X is not Hausdorff.

(2) Determine whether the following spaces are homeomorphic. Make sure that you provide a proof:
  (a) The unit interval $I := [0,1]$ and the circle S^1.
  (b) The open unit interval $(0,1)$ and the closed unit interval $[0,1]$.
  (c) The open unit interval $(0,1)$ and the positive real line $(0,\infty)$.

(3) Suppose that a group G acts continuously on a space X in the sense that for all $g \in G$ the map,
$$X \to X, x \mapsto g \cdot x$$
is continuous and let X/G denote the space consisting of all the orbits of this group action with its natural quotient topology. Prove that the quotient map,
$$X \to X/G$$
is an open mapping.

(4) Consider the equivalence relation on $X := \mathbb{R}^2$ given by,
$$x \sim y \Leftrightarrow \exists \lambda > 0 : y = \lambda x.$$
Prove that there are no disjoint closed sets in the quotient space $X/\sim$ and hence that the space $X/\sim$ is normal.

(5) Using the Seifert-Van Kampen Theorem or otherwise, compute the fundamental group of the real projective plane $\mathbb{R}P^2$. Recall that $\mathbb{R}P^2$ is homeomorphic to a disk in which antipodal points of the boundary are identified.

(6) Give a complete proof of Urysohn's Lemma.

(7) Let $\rho : \widetilde{X} \to X$ be a covering space mapping a basepoint $\tilde{x}_0 \in \widetilde{X}$ to a basepoint $x_0 \in X$. Prove that any path $\gamma : [0,1] \to X$ with initial point x_0 has a lift $\gamma : [0,1] \to \widetilde{X}$ starting at $\tilde{x}_0$,
$$\rho \circ \tilde{\gamma} = \gamma.$$

(8) Using Mayer-Vietoris or otherwise, compute the homology of the spheres S^n for $n \geq 0$.