TOPOLOGY PRELIMINARY EXAM
AUGUST 2022

Endeavor to provide details for any arguments made during the course of a proof that are as complete as possible. If any major theorem is used in any argument, give a precise statement of that theorem.

There are eight questions on the exam. You will be graded on your seven best answers.

(1) Consider the real line $\mathbb{R}$ and its quotient $\mathbb{R}/\mathbb{Q}$ by the subgroup $\mathbb{Q}$ of rational numbers:
$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$
Prove that $\mathbb{R}/\mathbb{Q}$ has the trivial topology.

(2) Determine whether the following spaces are homeomorphic. Make sure that you provide a proof:
(a) The spaces
$$X := \{x \in \mathbb{R} : |x| > 0\} \quad \text{and} \quad Y := \{x \in \mathbb{R} : x > 0\}.$$
(b) The spaces
$$X := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\} \quad \text{and} \quad Y := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$
(c) The spaces
$$X := \mathbb{R}^2 \quad \text{and} \quad Y := \mathbb{R}^2 - \{0\}.$$

(3) Consider the projection map
$$\pi : \mathbb{R}^2 \to \mathbb{R}, \quad (x, y) \mapsto x.$$
Prove that this is *not* a closed mapping (taking closed sets to closed sets).

(4) Consider the interval $X := [0, 1]$ and the subspace
$$A := \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$$
of X. Consider the quotient X/A, that is the quotient of X by the relation
$$x \sim y \Leftrightarrow (x \in A \wedge y \in A).$$
Prove that X/A is *not* Hausdorff.

(5) Using the Seifert-Van Kampen Theorem, compute the fundamental group of the Klein bottle. Recall that the Klein bottle is obtained from a cylinder by gluing the ends using a reflection.

(6) Give a complete proof of Urysohn's Lemma.

(7) Let $\rho : \widetilde{X} \to X$ be a covering space mapping a basepoint $\tilde{x}_0 \in \widetilde{X}$ to a basepoint $x_0 \in X$. Prove that any path $\gamma : [0, 1] \to X$ with initial point x_0 has a lift $\tilde{\gamma} : [0, 1] \to \widetilde{X}$ starting at $\tilde{x}_0$,
$$\rho \circ \tilde{\gamma} = \gamma.$$

(8) Using Mayer-Vietoris or otherwise, compute the homology of the spheres S^n for $n \geq 0$.