TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
August 2023

WORK ALL PROBLEMS. ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. (a) Define the notion of a basis of a topological space.
   (b) Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$ and let $\mathcal{T}'$ be the topology defined by the basis $\{(a,b] \mid a,b \in \mathbb{R}, a < b\}$. Prove that $\mathcal{T}'$ is finer than $\mathcal{T}$ and does not coincide with $\mathcal{T}$.

2. (a) Let X, Y be topological spaces. Prove that $f : X \to Y$ is continuous if and only if for every subset A of X, one has
   $$f(\overline{A}) \subset \overline{f(A)}.$$
   Here $\overline{A}$ denotes the closure of the set A.
   (b) Let X_α, $\alpha \in A$ be a family of topological spaces, and let Y be a topological space. Endow $\prod_\alpha X_\alpha$ with the product topology, and let π_α be the projection of $\prod_\alpha X_\alpha$ onto X_α, $\pi_\alpha((x_\beta)_\beta) = x_\alpha$. Show that a function $f : Y \to \prod_\alpha X_\alpha$ is continuous if and only if $f_\alpha = \pi_\alpha \circ f$ is continuous for every α.

3. Let $\mathcal{U}$ be an open covering of the compact metric space X. Prove that there exists $\delta > 0$ such that for each subset of X having diameter less than δ, there is an element of $\mathcal{U}$ containing it.

4. (a) State Urysohn's Lemma.
   (b) State the Tietze Extension Theorem.
   (c) State the Seifert-van Kampen Theorem.

5. (a) Define the fundamental group of a path connected space.
   (b) Let X be a path connected space and x_0 and x_1 two points in X. Prove that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

6. Compute the fundamental group of the genus 2 surface.

7. (a) Compute the homology groups with integer coefficients of the projective plane.
   (b) Compute the homology groups with real coefficients of the 2-dimensional torus. Also, compute its Euler characteristic.

8. Let n be a positive integer and let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, and consider the map
$$f : S^1 \to S^1, f(z) = z^n.$$
   (a) Turn the circles that are the domain of f and the range of f into Δ-complexes so that f becomes a Δ-map.
   (b) Compute the group homomorphisms induced by f on the zeroth, first, and second homology groups with integer coefficients of the circle.