TOPOLOGY DOCTORAL PRELIMINARY EXAMINATION
May 2023

WORK ALL PROBLEMS. ALL SPACES UNDER CONSIDERATION ARE HAUSDORFF (T_2). GIVE AS COMPLETE ARGUMENTS FOR PROOFS AND DESCRIPTIONS OF EXAMPLES AS POSSIBLE. IF ANY MAJOR THEOREM IS USED IN ANY ARGUMENT, GIVE A PRECISE STATEMENT OF THE THEOREM.

1. (a) Prove that the image through a continuous map of a path connected space is path connected.
   (b) Prove that the image through a continuous map of a compact set is compact.
   (c) Prove that the union of two connected spaces that have a common point is connected.
   (d) Prove that the product of two compact spaces is compact.

2. Decide, and explain your answer, which of the following spaces are homeomorphic to each other and which are not (each has the standard subspace topology):
   (a) the interval (-1,1);
   (b) the interval $(0, 1)$;
   (c) the sphere $S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + x_3^2 = 1\}$;
   (d) the sphere $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \,|\, x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$.

3. Let X be a locally compact Hausdorff space. Prove that there exists a compact space Y containing X such that $Y \backslash X$ consists of one point.

4. State and prove the Path Lifting Lemma.

5. (a) Show that every continuous map of a circle into a line maps some pair of diametrically opposite points to the same point.
   (b) Show that every continuous map from the 2-dimensional sphere to the plane maps some pair of antipodal points to the same point.

6. (a) Compute the fundamental group of the wedge of two circles.
   (b) Compute the fundamental group of the 2-dimensional torus.
   (You may assume that the fundamental group of the circle is known).

7. (a) Compute the simplicial homology with integer coefficients of the Klein bottle.
   (b) Compute the simplicial homology with real coefficients of the Klein bottle, and compute its Euler characteristic.

8. Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ and consider the continuous map

$$f : S^1 \times S^1 \to S^1 \times S^1, \quad (z, w) \mapsto (z^2, w^3).$$

   (a) Put on the domain and on the range of f structures of Δ-complexes so that f is a Δ-map.
   (b) Compute the group homomorphisms induced by f on the zeroth, first, and second homology groups with integer coefficients of the torus.