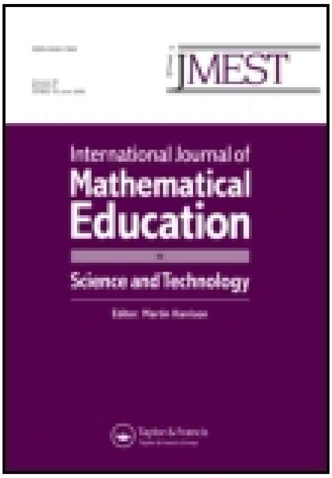

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A professional development model for middle school teachers of mathematics

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Teacher professional development activities in the USA take many forms from half-day workshops that focus on particular topics or classroom techniques to long term course work that offers university level credit. With few exceptions, the primary goal of such activities is to enhance the teachers' classroom effectiveness and improve student achievement. In this article, we describe a professional development model that strives to provide middle school mathematics teachers with a deep understanding of the mathematics they teach, and our attempts to measure its influence on their mathematics content knowledge.

Keywords: professional development; mathematics knowledge for teaching; algebraic structure; measurement

1. Introduction

In the late 1980s the mathematics community in the USA began to place much attention on the mathematics preparation of primary and secondary school mathematics teachers. The Mathematical Association of America put out its call for change [1] and the National Council of Teachers of Mathematics (NCTM) announced its curriculum and evaluation standards for school mathematics [2] and its professional standards for the teaching of mathematics at the primary and secondary levels [3]. At the same time, a discouraging report on the preparation of USA elementary school mathematics teachers came out of a major conference held at the University of Chicago [4]. Inspired by these events, we initiated a project to reform the mathematics courses required of all students at our university who were preparing to teach in the primary grades. Our efforts resulted in a three course mathematics sequence required of all such students [5,6].

The results of the Trends in International Mathematics and Science Study [7] released in 1999 fueled a greater sense of urgency in the USA education community when it showed that USA eighth grade students demonstrated significantly lower proficiency in mathematics than students from Singapore, The Republic of Korea, Chinese Taipei, Hong Kong, Japan, Canada, Australia, and seven European countries. The same year Ma [8] published the results of her extensive study comparing the characteristics and practices of middle school mathematics teachers in the USA with those of middle school mathematics teachers in China. Ma found that

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the Chinese teachers exhibited a much deeper conceptual understanding of the mathematics they were teaching than did their USA counterparts. She emphasized the need for middle school mathematics teachers to have a deep conceptual understanding of the elementary mathematics taught in middle school. However, a recently released report from the National Academy of Sciences paints a rather bleak picture of elementary mathematics education in the USA. The report contends that more than half of the teachers of grades 5–8 (middle school) in the USA neither majored in mathematics nor are certified to teach mathematics [9, p. 6]. These findings contributed to an increased interest in the enhancement of the professional development opportunities for the middle school mathematics teachers.

There is a long history of formal professional development programmes targeting primary and secondary teachers in the USA. The publications by Loucks-Horsley et al. [10] and Wei et al. [11] contain a summary of the development of, rationale for, and current state of such professional development programmes in the USA. Typically, all school teachers in the USA are required to participate in professional development activities in order to maintain their teacher certifications, but standards for such participation vary from state to state, and even from one school district to another within states. However, approximately 90% of all school teachers in the USA regularly participate in professional development activities [11]. Typically such activities involve attendance at half-day, or full-day workshops that provide lecture or presentations that promote specific classroom suggestions or interventions [12]. We are doubtful whether such professional development activities can provide these teachers with the deep conceptual understanding of middle school mathematics as identified by Ma [8] and recommended by the NCTM [13].

With funding from the National Science Foundation (NSF), we embarked on a project to develop a different professional development model for middle school teachers of mathematics: the West Texas Middle School Math Partnership (WTMSMP). The cornerstone of our model is a collection of three intensive, 2 week summer mathematics courses offered at four institutions of higher education to middle school teachers of mathematics in our region. The purpose of this article is to examine the influence this model had on the participants in its first 2 years of implementation. We now proceed with a description of the first two of these courses.

2. The courses

2.1. *Theoretical foundation*

We believe that it is necessary for the teacher of middle school mathematics to have a deep conceptual understanding of the mathematics taught in the middle grades. However, such knowledge alone is not sufficient. The question of just what skills and knowledge are required of an effective middle school teacher was posed by Shulman [14,15] in the late 1980s and has been a topic of intense research ever since. Shulman referred to such knowledge as *pedagogical content knowledge* (PCK). Deborah Ball [16], working with various colleagues, has focused on knowledge specific to teaching mathematics and expanded Shulman's ideas to include *mathematical knowledge for teaching* (MKT).

Much of PCK consists of practical knowledge such as knowledge of pedagogy, content, classroom management, curriculum, student learning, development. In addition to these types of knowledge and skills, MKT contains *specialized content* *knowledge* (SCK) [16]. SCK tends to be more theoretical and conceptual; for example, with focus on the teachers' ability to detect and correct student mathematical misconceptions, and their ability to assess the validity and generalizability of a student's non-standard approach to solving problems encountered in the middle school mathematics curriculum [17, p. 6].

Ball has assigned specific knowledge and skill criteria to Ma's call for middle school teachers to have a deep conceptual understanding of the mathematics taught in middle school. It is this that has driven the development and delivery of our mathematics courses.

2.2. *Course 1*

Course 1, titled 'Integers and fractions: An investigation into the algebraic structure of our numbers', begins with the natural numbers from the point view of Bertrand Russell's [18] idea of number classes, and addition is defined by combining (disjoint) number classes. The commutative and associative properties follow naturally. Zero is defined to be the integer with number class the empty set, then by postulating the existence of additive inverses, the structure of the group of integers ensues. Exercises involve teachers discussing common student misconceptions and possible ways of addressing them. Also, teachers are asked to create novel classroom activities and concrete models demonstrating the various concepts, and they are asked to evaluate each such activity and model. For example, the teachers are asked to provide such a model that demonstrates the meaning of $5+(-2)=3$ and $2+(-5)=-3$. In one class, a group came up with the following model: Let 5 represent 5 dogs, -5 represent 5 bones, 2 represent 2 dogs and -2 represent 2 bones, with addition being the obvious combination of dogs and bones. After a brief discussion, another group of teachers decided this model was flawed because combining 5 dogs and 5 bones, results in 5 dogs and *no* bones. Immediately, the first group saw the problem and suggested that it could be corrected by letting the positive integer represent the number of hungry dogs. With that change, the model was deemed by all to be acceptable for use in the classroom.

Multiplication of positive integers is defined as repeated addition and extended axiomatically to the set of all negative integers (additive inverses of positive integers), thus developing the ring structure of the integers. The multiplicative inverse of an integer is postulated. Addition and multiplication involving integers and multiplicative inverses of integers are axiomatically defined to preserve all the existing ring structure, leading the rational number field. Again, teachers are asked to provide classroom activities and concrete models for all these operations.

Finally, the least upper bound principal is introduced, leading to the existence of irrational numbers. All of this is done using only concepts introduced in middle school mathematics classes.

2.3. *Course 2*

Course 2 is titled 'Size in theory and practice' and covers topics from geometry with emphasis on measure (size) of sets in zero, one, two, and three dimensions. The size of a zero-dimensional set (a finite set of points) is defined to be the number of points in the set.

The basic one-dimensional set is defined to be the set swept out by a zero-dimension set of size 1 (a set with one point) translated a finite distance in a straight line (a line segment). The size of such a set is defined to be the measure of the distance the point is translated. Different units of linear measure are discussed and compared. The distance formula for line segments in the plane is defined *via* the Pythagorean Theorem (gotten *via* an algebraic argument, area not yet having been discussed). Perimeters of polygons are discussed. This chapter ends with the definition of the number π and the circumference of circles.

The basic two-dimensional set is defined by the region swept out by a line segment that is perpendicularly translated a given distance (a rectangle). The size (area) of such a region is defined to be the size (length) of the line segment times the perpendicular translation distance. The area formulae for all the polygons typically considered in middle school are derived from this basic concept. The area of a circle is defined as the limiting value of the areas of the inscribed regular polygons, and the area formula for the circle is obtained.

The basic three-dimensional set is defined by the region swept out by a rectangle that is perpendicularly translated a given distance (rectangular right prism). The size (volume) of such a region is defined to be the size (area) of the rectangle times the perpendicular translation distance. Again, all the usual volume formulas for prisms are obtained. Cavalier's principle is used to find formulae for the sizes of slant prisms and cylinders, and parallelograms and triangles with same height and base. Much effort goes into providing a rigorous derivation of the volume formula for a pyramid. The volume of the sphere is gotten using a very clever argument attributed to Archimedes [19].

Course 2 ends with a discussion of fractal dimension, with the teachers working through the online examples provided by Connors [20]. As with Course 1, Course 2 contains many exercises in which teachers are asked to construct activities and models suitable for use in their middle school classrooms.

At the time of this writing, 64 in-service middle school mathematics teachers have completed Courses 1 and 2. The influence of these courses on the teachers' *mathematics content knowledge* (MCK), as well as their MKT, is the subject for the remainder of this article.

3. The influence on MCK and MKT

3.1. *Participants*

Of the original 65 WTMSMP participating middle school mathematics teachers, 83.1% were women ($n=54$) and 15.4% were men ($n=10$). One person failed to report gender. In Year 1 (summer 2009), participants reported an average of 10.46 years (SD 7.35) ranging from 1 to 32 years. When asked about years of experience teaching mathematics, participants reported teaching mathematics for an average of 9.26 years (SD $=6.59$), ranging from 0 to 27 years.

Mathematical background was determined in Year 1 by asking participants to identify mathematics courses of a certain type taken in college (e.g. college algebra, pre-calculus, calculus, statistics, and differential equations). Participants were assigned one point for each course type taken and these points were summed. An average total sum of 3.63 (SD $=2.50$) was calculated, with range of 0 to 8.

At the start of Year 2 (summer 2010), three participants stepped out of the project. A fourth withdrew prior to the completion of the second course. Due to some submission of incomplete measures at certain time points, the analyses of the MCK data included 60 participants and analyses of MKT data included 58 participants for the Algebra and Geometry tests and 59 participants for the Number Concepts (NC) and Operations test.

In the spring of 2010, a solicitation for comparison teachers yielded 14 volunteer middle level teachers, 1 man and 13 women. Comparison group teachers reported teaching an average of 10.61 years (SD = 8.32) and teaching mathematics an average of 7.89 years (SD = 5.41). Comparison teachers were located in each of the three regions of the WTMSMP project.

3.2. *Instruments*

Teachers' mathematics knowledge for teaching was assessed using the MKT scales developed for the Study of Instructional Improvement and Learning Mathematics for Teaching projects located at the University of Michigan. These scales, developed using data from teachers and mathematicians, include measures that assess knowledge for teaching NC and Operations, Algebra, and Geometry [21]. Validity studies evaluating the MKT scales have included cognitive interviews [22], unidimensional and multidimensional Item Response Theory (IRT) mapping [23] and associations between the MKT scales and student outcomes as well as mathematics instruction [24]. Although structural evidence that supports differentiation between the mathematics knowledge specific to teachers and common mathematics knowledge is lacking [25], higher teacher MKT scores have been found to be positively related to higher quality mathematics instruction [24,26] and gains in student learning [27]. The MKT tests include items that range from easy to difficult in level, with the expectation that difficult items will be answered correctly by only a small number of test takers. Thus, an IRT score of 0 indicates that a participant solved about 50% of the problems correctly; however, a score of 0 does not necessarily indicate that the participant scored in the average range as the test is not norm referenced. The results only provide information concerning how well participants performed on the present administrations of the MKT.

As there was no attempt to design the course materials in any direct alignment with the MKT measures, the WTMSMP researchers created a mathematics content knowledge (MCK) measure specifically aligned with the geometry and measure content of Course 2. The MCK instrument consists of 36 items, to each of which the teachers are asked to respond 'true', 'false', or 'I don't know'. Items were created by the mathematician who developed the WTMSMP course curriculum and evaluated by a second mathematician who has contributed to course development and instruction. Correlations between the MCK post-test and the three MKT post-tests for Year 2 were predominately moderate ranging from $r = 0.61$ ($p < 0.001$) with the Geometry MKT measure to $r = 0.76$ ($p < 0.001$) with the NC scale. These results seem to reflect that although the instruments share content, they also assess different constructs. Additionally, the correlation between the MCK and teachers' total number of years teaching was statistically non-significant and small ($r = -0.10$, $p < 0.47$); whereas, the correlation between the MCK and teachers' self-reported mathematical background was statistically significant and moderate ($r = 0.50$, $p < 0.001$). These findings further support that the focus of the MCK is on the assessment of mathematical skill rather than teaching.

3.3. *Methodology*

Participants completed parallel versions of all three MKT measures: the 2007 Middle School NC scale comprised of 30 items for form A and 32 for form B, the 2005 Middle School Geometry (Ge) scale comprised of 19 items for form A and 23 for form B, and the Middle School Algebra (Al) scale comprised of 33 items for forms A and B. All three measures were administered at four time points: pre- and post-Course 1 in summer 2009 and pre- and post-Course 2 in summer 2010. For the algebra and geometry scales, version A was administered at the pre-test and version B at the post-test. However, administration of parallel versions was counterbalanced (half received version A and half version B at the pre-test followed by the appropriate corresponding test at the post-test) for the NC measure, as evidence for statistical equivalency of the two parallel versions was reported to be weaker than observed for the other measures [28]. In summer 2010, all participants were given the MCK measure in pre- and post-Course 2 format. All measures were completed and submitted electronically.

Raw MKT scores were converted to IRT scores given in standard deviation units with tables provided by the test developers. A term for using the MKT tests is that MKT scores not be discussed as raw frequencies or number correct. Thus, we utilized IRT pre- and post-test gain scores in this study.

The MCK items were scored as follows: a score of 0 was given to a wrong true or false answer, a score of 1 was given to an 'I don't know' response, and a score of 2 was awarded to a correct true or false answer. We view a score of 0 as an indication of a misconception and a score of 1 as simply indicating a lack of knowledge. Raw scores were divided by 36, thus providing an indication of the participants' content knowledge level on a scale from 0 to 2.

3.4. *Results*

Profile analyses [29] were used to evaluate the parallelism, equality of levels, and flatness of profiles for each of the MKT measures. Participants who did not take mathematics beyond college algebra were included in one group ($n = 15$) and those that did were included in the second group ($n = 42$). Analyses of each measure were conducted with the scores of the four time points treated as multiple dependent variables. Paired t-tests were then used to compare the Year 1 MKT pre-tests and Year 2 post-tests and to compare the Year 2 MCK pre-test scores and Year 2 MCK post-test scores, as the MCK measure did not exist in Year 1. Finally, analyses of the difference between WTMSMP participants' Year 2 MKT post-tests and comparison participants' spring 2010 MKT scores were conducted. Prior to conducting all analyses, descriptive statistics were computed and statistical assumptions for subsequent analyses evaluated.

3.4.1. *Differences in growth based on mathematical background*

Profile analysis results for each MKT measure were similar. The profiles of group 1 and group 2 teachers were similar or parallel. That is, there was no interaction observed between group and time. Additionally, the profile of the two groups combined was predominately flat. This means that the slopes for the combined groups between each segment (i.e. time one and time two, time two and time three, and time three and time four) did not differ significantly from zero. Finally, a significant difference was observed between groups for each MKT measure. The teachers who had taken mathematics courses beyond college algebra did score, on average, higher on the collected set of each measure. Table 1 presents the statistics for each research question as well as the estimates of observed power, which were somewhat low to suggest the small sample size may have influenced the ability to determine statistical significance.

3.4.2. *Overall growth of participants*

Paired sample t-tests for each MKT measure (pre-test Year 1 and post-test Year 2) and the MCK measure (pre-test and post-test) revealed a significant increase for Geometry ($t(57)=2.38$, $p=0.02$) and the MCK measure ($t(59)=11.84$, $p<0.01$). The average total IRT score gains from the pre-test Year 1 to post-test Year 2 and the average knowledge level gains from the MCK measure in Year 2 are presented in Table 2. The comparable IRT and MCK median data are presented in Table 3.

Table 1. Profile analysis results.

Research question	MKT measure	F	df	Significance	Partial eta squared	Observed power
Parallelism	NC	1.32	3, 53	0.28	0.07	0.33
(Wilks' criterion)	Algebra	0.36	3, 52	0.78	0.02	0.12
	Geometry	1.34	3, 52	0.27	0.07	0.34
Flatness	NC	0.89	3, 53	0.45	0.05	0.23
(Hotelling's criterion)	Algebra	0.83	3, 52	0.48	0.05	0.22
	Geometry	1.01	3, 52	0.40	0.06	0.26
Equality of levels	NC	8.40	1, 55	0.01	0.13	0.81
	Algebra	12.93	1, 54	<0.01	0.19	0.94
	Geometry	5.90	1, 54	0.02	0.10	0.67

Table 2. Average MKT IRT gain from pre-test Year 1 to post-test Year 2, MCK average gain from pre-test Year 2 to post-test Year 2.

	MKT-NC	MKT-Al	MKT-Ge	MCK-Ge
Average gain	0.17	0.12	0.17	0.29
SD	0.72	0.65	0.56	0.19

Table 3. MKT IRT median score pre-test Year 1 and post-test Year 2, MCK median score pre-test Year 2 and post-test Year 2.

MKT-NC		MKT-Al		MKT-Ge		MCK-Ge	
Pre-1	Post-2	Pre-1	Post-2	Pre-1	Post-2	Pre-2	Post-2
−0.15	0.02	−0.18	−0.09	0.53	0.97	1.39	1.75

Although comparisons for NC and Algebra revealed increases, these were not statistically significant.

3.4.3. *Differences between participants and non-participants*

The 14 comparison group teachers were matched to 14 WTMSMP participants based on years teaching mathematics, years teaching, and region. Although exact matches could not be made across all three variables, years teaching mathematics was closely matched for each pair. Paired sample t-tests were conducted for each of the MKT measures, including NC, Algebra, and Geometry to assess differences between comparison teachers' spring 2010 performance on MKT measures and participant teachers' Year 2 post-test performance. Statistically significant differences were found for NC ($t(13) = 2.94$, $p = 0.01$) and Algebra ($t(13) = 2.88$, $p = 0.01$). These findings indicate that WTMSMP participants achieved higher scores on the NC and Algebra MKT measures than comparison group teachers.

3.5. *Discussion*

The aforementioned results suggest that the WTMSMP is progressing in its development of mathematics educators' mathematics conceptual knowledge and mathematics knowledge for teaching. Profile analyses revealed that the grand mean of all time points for each of the MKT measures differed depending on the participants' mathematical background. Those who had taken no more than a college algebra course performed significantly lower on each MKT measure than those who had taken courses beyond college algebra. However, the profiles of the two groups were parallel to suggest that they had the same patterns of gains over the four time points. This suggests that regardless of mathematical preparation, teachers were responding similarly to the WTMSMP content. Even though the flatness of profiles could not be rejected to suggest statistically significant growth over time, a review of plots as well as subsequent paired-samples t-tests indicates that the participants' MKT scores are increasing. Collectively, these results are promising and support that all participants are benefitting from the project. An increase in power, which will be achieved as a second cohort of the WTMSMP project will add to the sample size, may help in documenting statistical significance.

In response to the concern that the MKT measure's emphasis on teachers' use of mathematical knowledge in the classroom might limit the ability to evaluate WTMSMP participants' growth in conceptual knowledge for mathematics, the WTMSMP team members developed and administered an additional instrument based on conceptual understanding of WTMSMP course content. Although evaluation of the psychometric properties associated with the instrument is currently limited, the participants showed a statistically significant increase in their post-test scores in comparison to the pre-test. This indicates that participants' mathematical understanding is improving.

Finally, a comparison between WTMSMP participants and matched non-participating teachers further supports the influence of the WTMSMP project. The analysis of participants matched to mathematics teachers with similar teaching experience revealed that participants outperformed non-participants on the MKT tests of NC and Algebra.

WTMSMP participant growth in mathematics knowledge for teaching may be slow, but aspects of MKT are complicated. The MKT measures do not solely assess teachers' mathematical knowledge. Instead, they measure how teachers use their mathematical knowledge in the classroom. One premise of the WTMSMP is that teaching deeper conceptual knowledge of mathematics to teachers will change the way that teachers teach through their ability to develop more meaningful examples, react flexibly to student questions, and easily identify student misconceptions and errors. The WTMSMP participants need time and support to transfer their new conceptual knowledge learned in the summer courses to their classrooms. This application may require more support than what is received only in summer sessions. The project also includes spring conferences and online social networking; however, these aspects are currently not as well developed as the course content.

4. Conclusions

An emphasis on quality teacher development has emerged in response to the poor mathematics performance of American youth in international comparisons. Despite clear recommendations generated from empirical study for the design of mathematics teacher development [10], most educators participate in time- and content-limited day-long workshops that are not formally evaluated for their effect on mathematics teacher and student outcomes, such as increased knowledge and skill for both teachers and students. WTMSMP was funded by the NSF to address and study this issue through the development of mathematics coursework designed and implemented specifically by mathematicians to facilitate middle level mathematics teachers' conceptual knowledge for mathematics. After its first 2 years of implementation, data analyses indicate that WTMSMP participants' mathematics knowledge for teaching and conceptual knowledge for mathematics are increasing regardless of the initial skill level of teachers. Additionally, evidence suggests that participants may have the knowledge to use their mathematics knowledge in their classrooms in a more effective manner than similarly matched peers. The present findings support that the development of mathematics knowledge for teaching is a process that requires intense study over time.

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References

[1] Mathematical Association of America, *A Call for Change: Recommendations for the Mathematical Preparation of Mathematics*, MAA, Washington, DC, 1991.
[2] National Council of Teachers of Mathematics, *The Curriculum and Evaluation Standards for School Mathematics*, NCTM, Reston, VA, 1989.
[3] National Council of Teachers of Mathematics, *Professional Standards for Teaching Mathematics*, NCTM, Reston, VA, 1991.

[4] B. Cipra, *On the mathematical preparation elementary school teachers*, Report on a Conference Held at the University of Chicago, National Science Foundation, University of Chicago, 1991.

[5] G. Harris and D. Tarwater, *The nine hour-hour mathematics sequence for the general pre-service elementary teacher at Texas Tech University*, PRIMUS VII(4) (1997), pp. 341–358.

[6] G. Harris and P. Schovanec, *College mathematics for elementary school teachers: A programme model?* New Zeal. J. Math. 32(supplementary issue) (2003), pp. 97–105.

[7] National Centre for Education Statistics, *Trends in International Mathematics and Science Study* (1999). Available at http://nces.ed.gov/timss/results.asp

[8] L. Ma, *Knowing and Teaching Elementary Mathematics: Teacher's Understanding of Fundamental Mathematics in China and the United States*, Lawrence Erlbaum Associates, Mahwah, NJ, 1999.

[9] A. Arrison, *Rapporteur, Rising Above the Gathering Storm Two Years Later: Accelerating the Progress Toward a Brighter Economic Future*, National Academies Press (2009). Available at http://books.nap.edu/catalog.php?record_id = 12537#toc

[10] S. Loucks-Horsley, K. Stiles, S. Mundry, N. Love, and P. Hewson, *Designing Professional Development for Teachers of Science and Mathematics*, 3rd ed., Corwin, Thousand Oaks, CA, 2010.

[11] R. Wei, L. Darling-Hannond, A. Andree, N. Richardson, and S. Orphanos, *Professional Learning in the Learning Profession: A Status Report On Teacher Development in the US and Abroad*, National Staff Development Council, Oxford, OH, 2009.

[12] T. Corcoran, *Helping teachers teach well: Transforming professional development*, in *Teacher Change and Development*, P.R. Villia, ed., Nova Science Publishers, New York, 2006, pp. 1–10.

[13] National Council of Teachers of Mathematics, *Principles and Standards for School Mathematics*, NCTM, Reston, VA, 2000.

[14] L. Shulman, *Those who understand: Knowledge growth in teaching*, Educ. Res. 15(20) (1986), pp. 4–14.

[15] L. Shulman, *Knowledge and teaching: Foundations of the new reform*, Harv. Educ. Rev. 57 (1987), pp. 1–22.

[16] D. Ball, M.G. Thames, and G. Phelps, *Content knowledge for teaching: What makes it special?* J. Teach. Educ. 59(5) (2008), pp. 389–407.

[17] D. Ball, *What mathematical knowledge is needed for teaching mathematics?* Secretary's summit on Mathematics, US department of Education. Washington, D.C., 6 February, 2003. Available at http://www-personal.umich.edu/~dball/presentations/020603_MathSummit.pdf

[18] B. Russell, *Definition of number*, in *The World of Mathematics*, Vol. 1, Simon and Schuster, New York, 1956, pp. 537–543.

[19] Archimedes, *Book II. On the sphere and cylinder*, in *The Works of Archimedes*, T.L. Heath, ed., Dover, New York, 1912.

[20] M. Connors, *Exploring fractal dimensions of strictly self-similar fractals: cantor dust to the fractal skewed web*. 2009. Available at http://www.math.umass.edu/~mconnors/fractal/fractal.html.

[21] S.G. Schilling and H.C. Hill, *Assessing measures of mathematical knowledge for teaching: A validity argument approach*, Measurement 5 (2007), pp. 70–80.

[22] H.C. Hill, C. Dean, and I.M. Goffney, *Assessing elemental and structural validity: Data from teachers, non-teachers, and mathematicians*, Measurement 5 (2007), pp. 81–92.

[23] S.G. Schilling, *Generalizability and specificity of interpretive arguments: Observations inspired by the commentaries*, Measurement 5 (2007), pp. 211–216.

[24] H.C. Hill, D.L. Ball, M. Blunk, I.M. Goffney, and B. Rowan, *Validating the ecological assumption: The relationship of measure scores to classroom teaching and student learning*, Measurement 5 (2007), pp. 107–118.

[25] S.G. Schilling, M. Blunk, and H.C. Hill, *Test validation and the MKT measures: Generalizations and conclusions*, Measurement 5 (2007), pp. 118–128.
[26] H.C. Hill, M. Blunk, C.Y. Charalambous, J.M. Lewis, G.C. Phelps, L. Sleep, and D.L. Ball, *Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study*, Cognit. Instruct. 26 (2008), pp. 430–511.
[27] H.C. Hill, B. Rowan, and D.L. Ball, *Effects of teachers' mathematical knowledge for teaching on student achievement*, Am. Educ. Res. J. 42 (2005), pp. 371–406.
[28] H.C. Hill, Technical report on 2005 and 2006 middle school forms Mathematics Knowledge for Teaching (MKT) measures. Author, Ann Arbor MI, 2007.
[29] B.G. Tabachnick and L.S. Fidell, *Using Multivariate Statistics*, 5th ed., Pearson, Boston, MA, 2007.