Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 5, Number 1, pp. 87–105 (2010)
http://campus.mst.edu/adsa

Some Oscillation Results for Second-Order Functional Dynamic Equations

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Abstract

In this paper, we consider the second-order nonlinear dynamic equation

$$\big(p(t)x^{\Delta}(t)\big)^{\Delta} + f\left(t, x^{\sigma}(t), x^{\tau}(t), x^{\xi}(t)\right) = 0,$$

on a time scale $\mathbb{T}$. Our goal is to establish some new oscillation results for this equation. Here we assume that $\tau(t) \le t \le \xi(t)$ for all t and $\tau,\ \xi : \mathbb{T} \to \mathbb{T}$, and use the notation $x^{\tau}(t) = x(\tau(t)), x^{\sigma}(t) = x(\sigma(t))$ and $x^{\xi}(t) = x(\xi(t))$. We apply results from the theory of lower and upper solutions for related dynamic equations along with some additional estimates on positive solutions.

AMS Subject Classifications: 34K11, 39A10, 39A99.
Keywords: Oscillation, functional dynamic equations, time scale, Riccati transformation technique.

1 Introduction

In 1988 the theory of time scales was introduced by Stefan Hilger in his Ph.D. Thesis in order to unify continuous and discrete analysis (see [20]). Not only does this unify the theories of differential equations and difference equations, but it also extends these classical situations to cases "in between"– e.g., to so-called q-difference equations. Moreover, the theory can be applied to other different types of time scales. Since its introduction, many authors have expounded on various aspects of this new theory, and we refer specifically to the paper by Agarwal et al. [1] and the references cited

Received February 18, 2009; Accepted September 25, 2009
Communicated by Martin Bohner

therein. A book on the subject of time scales by Bohner and Peterson [6] summarizes and organizes much of time scale calculus.

In recent years, there has been an increasing interest in studying the oscillation and nonoscillation of solutions of dynamic equations on a time scale (i.e., a closed subset of the real line $\mathbb{R}$). This has lead to many attempts to harmonize the oscillation theory for the continuous and the discrete cases, to include them in one comprehensive theory, and to extend the results to more general time scales. We refer the reader to the papers [2–5, 8–18, 21–23] and the references cited therein. To illustrate some of the results we mention the work of Zhang and Shanliang [23] who considered the delay equation

$$x^{\Delta\Delta}(t) + q(t)f(x(t-\tau)) = 0, \quad t \in \mathbb{T}, \tag{1.1}$$

where $\tau \in \mathbb{R}$ and $t - \tau \in \mathbb{T}$, $f : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing, and $uf(u) > 0$ for $u \neq 0$. By using comparison theorems they proved that the oscillation of (1.1) is equivalent to the oscillation of the nonlinear dynamic equation

$$x^{\Delta\Delta}(t) + q(t)f(x^{\sigma}(t)) = 0, \quad t \in \mathbb{T} \tag{1.2}$$

and established some sufficient conditions for oscillation by applying the results established in [17] for (1.2). In [19], we extend this result to show that the oscillation of

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q(t)f(x^{\sigma}(t)) = 0$$

is equivalent to that of

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q(t)f(x(\tau(t))) = 0$$

when $\displaystyle\int_{t_0}^{\infty} \frac{1}{p(t)}\Delta t = \infty$, $\displaystyle\frac{\mu(t)}{p(t)}$ is bounded on $\mathbb{T}$, and

$$\tau(t) \leq \sigma(t) \text{ for all } t \quad \text{ or } \quad \tau(t) \geq \sigma(t) \text{ for all } t.$$

Since we are interested in the oscillatory and asymptotic behavior of solutions near infinity, we assume throughout this paper that our time scale is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}}$ by

$$[t_0, \infty)_{\mathbb{T}} := [t_0, \infty) \cap \mathbb{T}.$$

Our main interest is to consider the general nonlinear dynamic equation

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + f\left(t, x^{\sigma}(t), x^{\tau}(t), x^{\xi}(t)\right) = 0, \quad t \in [t_0, \infty)_{\mathbb{T}}, \tag{1.3}$$

where $f \in C(\mathbb{T} \times \mathbb{R}^3, \mathbb{R})$ and $p \in C_{rd}([t_0, \infty)_{\mathbb{T}}, (0, \infty)_{\mathbb{T}})$ satisfies

$$\int_{t_0}^{\infty} \frac{1}{p(t)}\Delta t = \infty, \quad t \in [t_0, \infty)_{\mathbb{T}}.$$

We shall assume that

$$\tau(t) \le t \le \sigma(t) \le \xi(t)$$

for all $t \in \mathbb{T}$ and that $\tau, \xi \in C_{rd}(\mathbb{T}, \mathbb{T})$. We assume also that τ and ξ satisfy

$$\lim_{t\to\infty} \tau(t) = \infty = \lim_{t\to\infty} \xi(t).$$

Our attention is restricted to those solutions $x(t)$ of (1.3) which exist on some half-line $[t_x, \infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)| : t > t_0\} > 0$ for any $t_0 \ge t_x$. A solution x of (1.3) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

We note that (1.3) in its general form includes several types of differential and difference equations with delay or advanced arguments or both In addition, different equations correspond to the choice of the time scale $\mathbb{T}$. For example, when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$, $\mu(t) = 0$, $f^\Delta(t) = f'(t)$, $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, and so (1.3) includes the nonlinear delay differential equation

$$(p(t)x'(t))' + f\left(t, x(t), x^\tau(t), x^\xi(t)\right) = 0.$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\mu(t) = 1$, $f^\Delta(t) = \Delta f(t)$, $\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$, and a special case of (1.3) is the nonlinear delay difference equation

$$\Delta\left(p(t)\Delta x(t)\right) + f\left(t, x(t+1), x^\tau(t), x^\xi(t)\right) = 0,$$

where Δ denotes the forward difference operator. Of course many more examples may be given, and we will illustrate some of these in the examples in Section 3.

As observed above, equation (1.3) includes the delay and advanced argument cases. Concerning the function $f = f(t, u, v, w)$ we will always assume that f satisfies the following condition (A):

(A) $f(t, u, v, w) = -f(t, -u, -v, -w)$ and $f(t, u, v, w) > 0$ if $u, v, w > 0$, $t \in \mathbb{T}$.

We begin by introducing the auxiliary functions

$$P(t,a) = \int_a^t \frac{\Delta s}{p(s)}, \quad \eta(t,a) = \frac{P(\tau(t),a)}{P(\sigma(t),a)}, \quad \text{and} \quad \nu(t,a) = \frac{P(\xi(t),a)}{P(\sigma(t),a)}, \tag{1.4}$$

where $a \in \mathbb{T}$. We may now establish the following result.

Lemma 1.1. *Let y be a solution of* (1.3) *which satisfies*

$$y(t) > 0, \quad y^\Delta(t) > 0, \quad \textit{and} \quad \left(p(t)y^\Delta(t)\right)^\Delta \le 0$$

for all $\xi(t) \geq \sigma(t) \geq t \geq \tau(t) \geq T \geq t_0$. *Then we have*

$$y^\tau(t) \geq \eta(t,T)y^\sigma(t), \quad t \geq \tau(t) \geq T$$

and

$$y^\xi(t) \leq \nu(t,T)y^\sigma(t), \quad \xi(t) \geq t \geq T.$$

Proof. For $t > T \geq t_0$ we have

$$\begin{aligned} y^\sigma(t) - y^\tau(t) &= \int_{\tau(t)}^{\sigma(t)} y^\Delta(s)\Delta s \\ &= \int_{\tau(t)}^{\sigma(t)} \frac{1}{p(s)} p(s) y^\Delta(s)\Delta s \\ &\leq p(\tau(t))y^\Delta(\tau(t)) \int_{\tau(t)}^{\sigma(t)} \frac{1}{p(s)}\Delta s \end{aligned}$$

which yields

$$y^\sigma(t) \leq y^\tau(t) + p(\tau(t))y^\Delta(t)P(\sigma(t),\tau(t)).$$

Dividing both sides of this inequality by $y^\tau(t)$ we obtain

$$\frac{y^\sigma(t)}{y^\tau(t)} \leq 1 + \frac{p(\tau(t))y^\Delta(\tau(t))}{y^\tau(t)} P(\sigma(t),\tau(t)). \tag{1.5}$$

Also we have

$$\begin{aligned} y^\tau(t) - y(T) &= \int_{T}^{\tau(t)} y^\Delta(s)\Delta s \\ &= \int_{T}^{\tau(t)} \frac{1}{p(s)} p(s) y^\Delta(s)\Delta s \\ &\geq p(\tau(t))y^\Delta(\tau(t)) \int_{T}^{\tau(t)} \frac{1}{p(s)}\Delta s \\ &= p(\tau(t))y^\Delta(\tau(t))P(\tau(t),T) \end{aligned}$$

and hence

$$y^\tau(t) \geq p(\tau(t))y^\Delta(\tau(t))P(\tau(t),T).$$

Therefore we have

$$\frac{p(\tau(t))y^\Delta(\tau(t))}{y^\tau(t)} \leq \frac{1}{P(\tau(t),T)}. \tag{1.6}$$

Therefore, (1.5) and (1.6) imply

$$\begin{aligned} \frac{y^\sigma(t)}{y^\tau(t)} &\leq 1 + \frac{p(\tau(t))y^\Delta(\tau(t))}{y^\tau(t)} P(\sigma(t),\tau(t)) \\ &\leq 1 + \frac{P(\sigma(t),\tau(t))}{P(\tau(t),T)} \\ &= \frac{P(\sigma(t),T)}{P(\tau(t),T)}. \end{aligned}$$

This gives us the desired result

$$y^{\tau}(t) > \eta(t,T)y^{\sigma}(t).$$

The proof of the second inequality is similar. For $T < t \le \sigma(t) \le \xi(t)$, we have

$$y^{\xi}(t) - y^{\sigma}(t) = \int_{\sigma(t)}^{\xi(t)} y^{\Delta}(s)\Delta s \le p(\sigma(t))y^{\Delta}(\sigma(t)) \int_{\sigma(t)}^{\xi(t)} \frac{\Delta s}{p(s)}$$

and so we have

$$\frac{y^{\xi}(t)}{y^{\sigma}(t)} \le 1 + \frac{p(\sigma(t))y^{\Delta}(\sigma(t))}{y^{\sigma}(t)} P(\xi(t),\sigma(t)). \tag{1.7}$$

Also we have

$$y^{\sigma}(t) \ge y(T) + p(\sigma(t))y^{\Delta}(\sigma(t))P(\sigma(t),T)$$

so that

$$\frac{y^{\sigma}(t)}{p(\sigma(t))y^{\Delta}(\sigma(t))} \ge \frac{y(T)}{p(\sigma(t))y^{\Delta}(\sigma(t))} + P(\sigma(t),T).$$

Hence, from (1.7) we have

$$\begin{aligned} \frac{y^{\xi}(t)}{y^{\sigma}(t)} &\le 1 + \frac{p(\sigma(t))y^{\Delta}\sigma(t)}{y^{\sigma}(t)} P(\xi(t),\sigma(t)) \\ &\le 1 + \frac{P(\xi(t),\sigma(t))}{P(\sigma(t),T)} \\ &= \nu(t,T) \end{aligned}$$

which yields the desired result

$$y^{\xi}(t) \le y^{\sigma}(t)\nu(t,T).$$

This completes the proof of the lemma. □

We shall also need the following lemma which is often referred to as the Riccati substitution technique.

Lemma 1.2 (See [11, Theorem 1]). *The linear equation*

$$Lx \equiv [p(t)x^{\Delta}(t)]^{\Delta} + q(t)x^{\sigma} = 0$$

is nonoscillatory if and only if there is a function z satisfying the Riccati dynamic inequality

$$z^{\Delta} + q(t) + \frac{z^2}{p(t)+\mu(t)z} \le 0 \tag{1.8}$$

with $p(t)+\mu(t)z(t) > 0$ for large t.

In order to prove our main results, we need a method of studying separated boundary value problems (SBVPs). Namely, we will define functions called upper and lower solutions that, not only imply the existence of a solution of a SBVP, but also provide bounds on the location of the solution. Consider the SBVP

$$\begin{aligned} -(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} &= f(t, x^{\sigma}), \quad t \in [a,b]^{\kappa^2}, && (1.9)\\ x(a) = A, \qquad & x(b) = B, && (1.10)\end{aligned}$$

where the functions $f \in C([a,b]^{\kappa^2} \times \mathbb{R}, \mathbb{R})$ and $p, q \in C_{rd}([a,b]^{\kappa^2})$ are such that $p(t) > 0$ and $q(t) \geq 0$ on $[a,b]^{\kappa^2}$. We define the set

$$\mathbb{D}_1 := \{x \in \mathbb{X} : x^{\Delta} \text{ is continuous and } px^{\Delta} \text{ is delta differentiable on } [a,b]^{\kappa} \text{ and } (px^{\Delta})^{\Delta} \text{ is rd-continuous on } [a,b]^{\kappa^2}\},$$

where the Banach space $\mathbb{X} = C([a,b])$ is equipped with the norm $\|\cdot\|$ defined by

$$\|x\| := \max_{t \in [a,b]_{\mathbb{T}}} |x(t)| \quad \text{for all} \quad x \in \mathbb{X}.$$

A function *x* is called a solution of the equation $-(p(t)y^{\Delta})^{\Delta} + q(t)y^{\sigma} = 0$ on $[a,b]^{\kappa^2}$ if $x \in \mathbb{D}_1$ and the equation $-(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = 0$ holds for all $t \in [a,b]^{\kappa^2}$. Next we define for any $u, v \in \mathbb{D}_1$ the sector $[u,v]_1$ by

$$[u,v]_1 := \{w \in \mathbb{D}_1 : u \leq w \leq v\}.$$

Definition 1.3 (See [7, Definition 6.1]). We call $\alpha \in \mathbb{D}_1$ a lower solution of the SBVP (1.9)–(1.10) on $[a,b]$ provided

$$-(p\alpha^{\Delta})^{\Delta}(t) + q(t)\alpha^{\sigma}(t) \leq f(t, \alpha^{\sigma}(t)) \quad \text{for all} \quad t \in [a,b]^{\kappa^2}$$

and

$$\alpha(a) \leq A, \quad \alpha(b) \leq B.$$

Similarly, $\beta \in \mathbb{D}_1$ is called an upper solution of the SBVP (1.9)-(1.10) on $[a,b]$ provided

$$-(p\beta^{\Delta})^{\Delta}(t) + q(t)\beta^{\sigma}(t) \geq f(t, \beta^{\sigma}(t)) \quad \text{for all} \quad t \in [a,b]^{\kappa^2}$$

and

$$\beta(a) \geq A, \quad \beta(b) \geq B.$$

Theorem 1.4 (See [7, Theorem 6.5]). *Assume that there exist a lower solution α and an upper solution β of the SBVP* (1.9)–(1.10) *such that*

$$\alpha(t) \leq \beta(t) \quad \textit{for all} \quad t \in [a,b].$$

Then the SBVP (1.9)–(1.10) *has a solution $x \in [\alpha, \beta]_1$ on $[a,b]$.*

The following is an extension of the previous theorem to $[a,\infty)_{\mathbb{T}}$.

Theorem 1.5. *Assume that there exists a lower solution α and an upper solution β of* (1.9) *with $\alpha(t) \le \beta(t)$ for all $t \in [a,\infty)_{\mathbb{T}}$. Then*

$$-(p(t)x^{\Delta})^{\Delta} + q(t)x^{\sigma} = f(t, x^{\sigma}) \tag{1.11}$$

has a solution x with $x(a) = A$ and $x \in [\alpha,\beta]_1$ on $[a,\infty)_{\mathbb{T}}$.

Proof. It follows from Theorem 1.4 that for each $n \ge 1$ there is a solution $x_n(t)$ of (1.9) on $[a,t_n]_{\mathbb{T}}$ with

$$x_n(a) = A, \quad x_n(t_n) = \beta(t_n), \quad \text{and} \quad \alpha(t) \le x_n(t) \le \beta(t)$$

on $[a,t_n]_{\mathbb{T}}$ where $\{t_n\}$ is such that $\lim\limits_{n\to\infty} t_n = \infty$. Thus, for any fixed $n \in \mathbb{N}$, $x_m(t)$ is a solution on $[a,t_n]_{\mathbb{T}}$ satisfying $\alpha(t) \le x_m(t) \le \beta(t)$ for all $m \ge n$. Hence, for $m \ge n$, the sequence $x_m(t)$ is pointwise bounded on $[a,t_n]_{\mathbb{T}}$.

We claim that $\left\{x_m^{\Delta}(t)\right\}$ is equicontinuous on $[a,t_N]_{\mathbb{T}}$ for any fixed $N \ge 1$. As q is right-dense continuous, it is regulated. It follows that q is bounded on $[a,t_N]_{\mathbb{T}}$, and so there exists a constant $Q_N > 0$ such that $|q(t)| \le Q_N$ for all $t \in [a,t_N]_{\mathbb{T}}$. Furthermore, since f is continuous and $x_m(t) \le \beta(t)$ for all $t \in [a,t_N]_{\mathbb{T}}$, there is a constant $K_N > 0$ such that

$$\begin{aligned}
\left|[p(t)x_m^{\Delta}(t)]^{\Delta}\right| &= |q(t)x_m^{\sigma}(t) - f(t,x_m^{\sigma}(t))| \\
&\le q(t)|x_m^{\sigma}(t)| + |f(t,x_m^{\sigma}(t))| \\
&\le Q_N|x_m^{\sigma}(t)| + K_N \\
&\le Q_N\|\beta\| + K_N \\
&=: M_N
\end{aligned}$$

for all $t \in [a,t_N]_{\mathbb{T}}$. It follows that

$$\begin{aligned}
\left|p(t)x_m^{\Delta}(t) - p(a)x_m^{\Delta}(a)\right| &= \int_a^t [p(s)x_m^{\Delta}(s)]^{\Delta}\,\Delta s \\
&\le M_N(t-a) \\
&\le M_N(t_N - a)
\end{aligned}$$

which gives

$$p(t)|x_m^{\Delta}(t)| = |p(t)x_m^{\Delta}(t)| \le |p(a)x_m^{\Delta}(a)| + |M_N(t_N - a)|.$$

Since $\{p(t)x_m(t)\}$ is uniformly bounded on $[a,t_N]_{\mathbb{T}}$ for all $m \ge N$, it follows that $|p(a)x_m^{\Delta}(a)| \le L_N$ for some $L_N > 0$ and all $m \ge N$. Consequently,

$$\left|p(t)x_m^{\Delta}(t)\right| \le L_N + |M_N(t_N - a)| := C_N,$$

and, immediately, we have

$$|x_m^\Delta(t)| \leq \frac{C_N}{p(t)} \leq C_N P_N \quad \text{for all} \quad t \in [a, t_N]_{\mathbb{T}}$$

since $1/p(t)$ is continuous on the compact interval $[a, t_N]_{\mathbb{T}}$. Consequently,

$$|x_m(t) - x_m(s)| = \left| \int_s^t x_m^\Delta(u)\ \Delta u \right| \leq C_N P_N |t - s| < \epsilon$$

for all $t, s \in [a, t_N]_{\mathbb{T}}$ provided $|t - s| < \delta = \dfrac{\epsilon}{C_N P_N}$. Hence the claim holds. So by Ascoli–Arzela and a standard diagonalization argument, $\{x_m(t)\}$ contains a subsequence which converges uniformly on all compact subintervals $[a, t_N]_{\mathbb{T}}$ of $[a, \infty)_{\mathbb{T}}$ to a solution $x(t)$, which is the desired solution of (1.11) with $x \in [\alpha, \beta]_1$ on $[a, \infty)_{\mathbb{T}}$. $\square$

2 Main Results

Our first result is for the case when $f(t, u, v, w)$ satisfies the following condition (C):

(C) For each fixed $t \in \mathbb{T}$ and $u, v > 0$, f is nonincreasing in w and for fixed $u, w > 0$, f is nondecreasing in v for $v > 0$ and for fixed $v, w > 0$ f is nondecreasing in u for $u > 0$.

Theorem 2.1. *Assume conditions* (*A*) *and* (*C*) *hold and let* $M > 0$. *Then any solution* x *of* (1.3) *with* $|x(t)| \leq M$ *is oscillatory in case*

$$\left| \int^\infty P(t, a) f\left(t, \alpha, \alpha\eta(t, a), M\nu(t, a)\right) \Delta t \right| = \infty, \tag{2.1}$$

for all $\alpha \neq 0$ *where* $\eta(t, a),\ \nu(t, a)$ *are given in* (1.4).

Proof. If not, let u be a bounded nonoscillatory solution which, in view of condition (A), we may assume satisfies

$$u(t) > 0,\ u(\tau(t)) > 0, \quad t \geq T \geq t_0.$$

Consequently,

$$[p(t)u^\Delta(t)]^\Delta = -f\left(t, u^\sigma(t), u^\tau(t), u^\xi(t)\right) < 0$$

for $t \geq T$ and so $p(t)u^\Delta(t)$ is decreasing for $t \geq T$.

We claim that $p(t)u^\Delta(t) > 0$ on $[T, \infty)_{\mathbb{T}}$. If not, there is a $t_1 \geq T$ such that $p(t_1)u^\Delta(t_1) < 0$. Then

$$p(t)u^\Delta(t) \leq p(t_1)u^\Delta(t_1), \quad t \in [t_1, \infty)_{\mathbb{T}},$$

and therefore

$$u^{\Delta}(t) \leq \frac{p(t_1)u^{\Delta}(t_1)}{p(t)}, \quad t \in [t_1, \infty)_{\mathbb{T}}.$$

Integrating, we obtain

$$u(t) - u(t_1) \leq \int_{t_1}^{t} u^{\Delta}(s)\Delta s \leq p(t_1)u^{\Delta}(t_1) \int_{t_1}^{t} \frac{\Delta s}{p(s)} \to -\infty$$

as $t \to \infty$, which implies $u(t)$ is eventually negative. This contradiction proves the claim. Hence, we conclude that for all $t \geq T$

$$u(t) > 0, \quad u(\tau(t)) > 0, \quad u^{\Delta}(t) > 0, \quad \big(p(t)u^{\Delta}(t)\big)^{\Delta} < 0.$$

From Lemma 1.1, we have

$$u(\tau(t)) \geq \eta(t,T)u^{\sigma}(t) \quad \text{and} \quad u(\xi(t)) \leq \nu(t,T)u^{\sigma}(t)$$

for $\xi(t) \geq \sigma(t) \geq t \geq \tau(t) \geq T$. By the monotonicity assumption on f we have

$$\begin{aligned} 0 &= [p(t)u^{\Delta}(t)]^{\Delta} + f\left(t, u^{\sigma}(t), u^{\tau}(t), u^{\xi}(t)\right) \\ &\geq [p(t)u^{\Delta}(t)]^{\Delta} + f\left(t, u^{\sigma}(t), \eta(t,T)u^{\sigma}(t), \nu(t,T)u^{\sigma}(t)\right) \end{aligned} \tag{2.2}$$

for $t \geq T$. Now, if we set

$$F(t, u^{\sigma}(t)) := f\left(t, u^{\sigma}(t), \eta(t,T)u^{\sigma}(t), \nu(t,T)u^{\sigma}(t)\right),$$

then (2.2) shows that $\beta(t) := u(t)$ is an upper solution for the dynamic equation $[p(t)u^{\Delta}]^{\Delta} + F(t, u^{\sigma}(t)) = 0$. Also, the constant function $\alpha(t) := u(T)$ satisfies $[p(t)\alpha^{\Delta}(t)]^{\Delta} + F(t, \alpha^{\sigma}(t)) \geq 0$, and so $\alpha(t)$ is a lower solution. Therefore, by Theorem 1.5, the BVP

$$[p(t)y^{\Delta}]^{\Delta} + F(t, y^{\sigma}(t)) = 0, \quad y(T) = u(T)$$

has a solution y with

$$u(T) \leq y(t) \leq u(t), \quad t \geq T.$$

It follows that $\lim\limits_{t\to\infty} p(t)y^{\Delta}(t) := L$ exists and is finite. If $L < 0$ or if $L = -\infty$, $p(t)y(t)$ would be eventually negative. Hence $L \geq 0$. Integration for $T < s < \tilde{T}$ implies

$$p(\tilde{T})y^{\Delta}(\tilde{T}) - p(s)y^{\Delta}(s) + \int_{s}^{\tilde{T}} F(r, y^{\sigma}(r))\Delta r = 0.$$

Letting $\tilde{T} \to \infty$ we obtain

$$p(s)y^{\Delta}(s) = L + \int_{s}^{\infty} F(r, y^{\sigma}(r))\Delta r \geq \int_{s}^{\infty} F(r, y^{\sigma}(r))\Delta r.$$

It follows that
$$y^{\Delta}(s) \geq \frac{1}{p(s)} \int_s^{\infty} F(r, y^{\sigma}(r))\Delta r.$$
Integrating again for $T < \tilde{t} < t$, we obtain
$$y(t) - y(\tilde{t}) = \int_{\tilde{t}}^{t} y^{\Delta}(s)\Delta s \geq \int_{\tilde{t}}^{t} \frac{1}{p(s)} \int_s^{\infty} F(r, y^{\sigma}(r))\Delta r \Delta s.$$
If we let
$$I_1(t) := \int_{\tilde{t}}^{t} \frac{1}{p(s)} \int_s^{\infty} F(r, y^{\sigma}(r))\Delta r \Delta s$$
and
$$I_2(t) := \int_{\tilde{t}}^{t} P(r, \tilde{t})F(r, y^{\sigma}(r))\Delta r + \int_t^{\infty} P(t, \tilde{t})F(r, y^{\sigma}(r))\Delta r,$$
then we have that $I_1(t) \geq I_2(t)$. Indeed, since
$$I_1^{\Delta}(t) = \frac{1}{p(t)} \int_t^{\infty} F(r, y^{\sigma}(r))\Delta r$$
and
$$I_2^{\Delta}(t) = P(t, \tilde{t})F(t, y^{\sigma}(t)) + \frac{1}{p(t)} \int_{\sigma(t)}^{\infty} F(r, y^{\sigma}(r))\Delta r + P(t, \tilde{t})(-F(t, y^{\sigma}(t)),$$
we have
$$[I_1(t) - I_2(t)]^{\Delta} = \frac{1}{p(t)} \int_t^{\sigma(t)} F(r, y^{\sigma}(r))\Delta r \geq 0.$$
So $I_1(t) - I_2(t)$ is increasing. Furthermore, since $I_1(\tilde{t}) = I_2(\tilde{t}) = 0$, the inequality follows. Consequently, for $t \geq \tilde{t} > T$, we have
$$\begin{aligned} y(t) - y(\tilde{t}) &= \int_{\tilde{t}}^{t} y^{\Delta}(s)\Delta s \\ &\geq \int_{\tilde{t}}^{t} \frac{1}{p(s)} \int_s^{\infty} F(r, y^{\sigma}(r))\Delta r \Delta s \\ &\geq \int_{\tilde{t}}^{t} P(r, \tilde{t})F(r, y^{\sigma}(r))\Delta r + \int_t^{\infty} P(t, \tilde{t})F(r, y^{\sigma}(r))\Delta r \qquad (2.3) \\ &\geq \int_{\tilde{t}}^{t} P(r, \tilde{t})F(r, y^{\sigma}(r))\Delta r. \end{aligned}$$
From (2.3) we have
$$y(t) \geq y(\tilde{t}) + \int_{\tilde{t}}^{t} P(r, \tilde{t})F(r, y^{\sigma}(r))\Delta r > \int_{\tilde{t}}^{t} P(r, \tilde{t})F(r, y^{\sigma}(r))\Delta r.$$

Since $y(t) \leq u(t) \leq M$ for some $M > 0$ and $\int_{\tilde{t}}^{t} P(r,\tilde{t})F(r,y^{\sigma}(r))\Delta r$ is an increasing function of t, it follows that

$$\int_{\tilde{t}}^{\infty} P(r,\tilde{t})F(r,y^{\sigma}(r))\Delta r < \infty.$$

By the monotonicity of f, we have

$$\int^{\infty} P(r,T)f\left(r,y(T),\eta(r,T)y(T),\nu(r,T)M\right)\Delta r < \infty.$$

By letting $\alpha = y(T)$, we obtain a contradiction to (2.1). This completes the proof. $\square$

The next result shows that a converse of Theorem 2.1 is true under an additional assumption.

Theorem 2.2. *Assume f satisfies conditions* (*A*) *and* (*C*) *and that*

$$\liminf_{t\to\infty} \eta(t,a) := k > 0 \quad \textit{and} \quad \limsup_{t\to\infty} \nu(t,a) := K < \infty, \tag{2.4}$$

where $\eta(t,a)$, $\nu(t,a)$ are defined in (1.4). *Also, assume that $P(\sigma(t),a)/P(t,a)$ is bounded and let $M > 0$. Then, if y is any nonoscillatory solution of*

$$(p(t)y^{\Delta})^{\Delta} + f\left(t,y^{\sigma}(t),y^{\tau}(t),y^{\xi}(t)\right) = 0 \tag{1.3}$$

with $|y(t)| \leq M$, it follows that

$$\left|\int^{\infty} P(\sigma(t),a)f\left(t,\alpha,\alpha\tilde{k},M\tilde{K}\right)\Delta t\right| < \infty$$

where $\tilde{k} < k$ and $\tilde{K} > K$.

Proof. Note that for any β

$$\left|\int^{\infty} P(\sigma(t),a)f(t,\beta,\dots,\beta)\Delta t\right| < \infty$$

if, and only if,

$$\left|\int^{\infty} P(t,a)f(t,\beta,\dots,\beta)\Delta t\right| < \infty$$

since $P(\sigma(t),a)/P(t,a)$ is bounded on $\mathbb{T}$.

Suppose (2.4) holds. Then for any $\epsilon > 0$ with $\epsilon < k$, there exists $\tilde{t} \geq t_0$ such that $\eta(t,a) > k-\epsilon =: \tilde{k}$ provided $t \geq \tilde{t}$ and there exists $\tilde{T} \geq t_0$ such that $\nu(t,a) \leq K+\epsilon =:$

$\tilde{K}$ provided $t \geq \tilde{T}$. Now assume (1.3) has a bounded nonoscillatory solution. Then by Theorem 2.1

$$\left| \int^{\infty} P(t,a) f\left(t, \alpha, \alpha\eta(t,a), M\nu(t,a)\right) \Delta t \right| < \infty$$

for all $\alpha \neq 0$. By the monotonicity assumption of f, we have

$$\int^{\infty} P(t,a) f\left(t, \alpha, \alpha\tilde{k}, M\tilde{K}\right) \Delta t < \infty$$

which proves the result. $\square$

The previous result says that condition (2.4) is sufficient in order to replace the auxiliary functions $\eta(t,a),\ \nu(t,a)$ given by (1.4) with upper bounds. Our next result gives a sufficient condition for

$$(p(t)y^{\Delta})^{\Delta} + f\left(t, y^{\sigma}(t), y^{\tau}(t), y^{\xi}(t)\right) = 0 \tag{1.3}$$

to have bounded nonoscillatory solutions.

Theorem 2.3. *Assume f satisfies conditions* (*A*) *and* (*C*)*. If*

$$\left| \int^{\infty} P(\sigma(t),a) f\left(t, \alpha, \alpha, \frac{\alpha}{2}\right) \Delta t \right| < \infty, \tag{2.5}$$

for all $\alpha \neq 0$ and

$$\frac{1}{p(t)} \int_t^{\infty} f\left(s, \alpha, \alpha, \frac{\alpha}{2}\right) \Delta s \leq M$$

for some $M > 0$ and all $t \geq a$, then (1.3) *has a bounded nonoscillatory solution.*

Proof. If (2.5) holds, assume to be specific that $\alpha > 0$ and let $0 < \beta < \alpha$. Choose $T \geq t_1 \in \mathbb{T}$ such that $\tau(t) \geq t_1$ for $t \geq T$ and such that

$$\int_T^{\infty} P(\sigma(t),a) f\left(t, \beta, \beta, \frac{\beta}{2},\right) \Delta t < \frac{\beta}{2}.$$

Define $y_0(t) \equiv \beta$ for $t \geq t_0$ and

$$y_{n+1}(t) = \begin{cases} \beta - \displaystyle\int_T^{\infty} (P(\sigma(s),a) - P(T,a)) f\left(s, y_n^{\sigma}(s), y_n^{\tau}(s), y_n^{\xi}(s)\right) \Delta s, & t < T, \\ \beta - \displaystyle\int_t^{\infty} (P(\sigma(s),a) - P(t,a)) f\left(s, y_n^{\sigma}(s), y_n^{\tau}(s), y_n^{\xi}(s)\right) \Delta s, & t \geq T. \end{cases}$$

Observe $t_1 \leq \tau(t) \leq t \leq \sigma(t) \leq \xi(t)$ for all $t \geq T$. We claim that

$$\frac{\beta}{2} \leq y_n(t) \leq \beta \quad t \geq T \quad \text{and} \quad \text{all } n \geq 0. \tag{2.6}$$

By construction the claim holds for $y_0(t)$. Notice that when $t < T$, we have $\tau(t) < T$ and so $y_n(\tau(t)) < \beta$ as $y^{\Delta}(t) \equiv 0$ for all $t \in \mathbb{T}$ less than T. Assume the inequality holds for $y_m(t)$, $1 \leq m \leq n$. Then for $t \geq T$

$$\begin{aligned} y_{m+1}(t) &= \beta - \int_t^{\infty} [P(\sigma(s),a) - P(t,a)]\, f\left(s, y_m^{\sigma}(s), y_m^{\tau}(s), y_m^{\xi}(s)\right) \Delta s \\ &\geq \beta - \int_t^{\infty} P(\sigma(s),a) f\left(s, y_m^{\sigma}(s), y_m^{\tau}(s), y_m^{\xi}(s)\right) \Delta s \\ &\geq \beta - \int_t^{\infty} P(\sigma(s),a) f\left(s, \beta, \beta, \frac{\beta}{2}\right) \Delta s \\ &> \frac{\beta}{2}. \end{aligned}$$

Furthermore, since $s \geq T$, we have $y_m^{\sigma}(s), y_m^{\tau}(s), y_m^{\xi}(s)$ are all positive. Hence by condition (A)

$$[P(\sigma(s),a) - P(t,a)]\, f\left(s, y_m^{\sigma}(s), y_m^{\tau}(s), y_m^{\xi}(s)\right) \geq 0$$

for $s \geq t \geq T$. Consequently, $y_{m+1}(t) \leq \beta$ for $t \geq T$. Therefore, by induction, (2.6) holds. It remains to show that the set $\{y_n(t)\}_{n=0}^{\infty}$ is equicontinuous. To do this, we show that $\left\{y_n^{\Delta}(t)\right\}_{n=0}^{\infty}$ is uniformly bounded. It follows that

$$\begin{aligned} \left|y_{n+1}^{\Delta}(t)\right| &= \bigg|0 - \Big[\int_t^{\infty} -P^{\Delta}(t,a) f\left(s, y_n^{\sigma}(s), y_n^{\tau}(s), y_n^{\xi}(s)\right) \Delta s \\ &\qquad - [P(\sigma(t),a) - P(\sigma(t),a)]\, f\left(t, y_n^{\sigma}(t), y_n^{\tau}(t), y_n^{\xi}(t)\right)\Big]\bigg| \\ &= \left|\int_t^{\infty} P^{\Delta}(t,a) f\left(s, y_n^{\sigma}(s), y_n^{\tau}(s), y_n^{\xi}(s)\right) \Delta s\right| \\ &= \left|\int_t^{\infty} \left(\int_a^t \frac{1}{p(s)} \Delta s\right)^{\Delta_t} f\left(s, y_n^{\sigma}(s), y_n^{\tau}(s), y_n^{\xi}(s)\right) \Delta s\right| \\ &= \left|\int_t^{\infty} \left(\int_a^t \left(\frac{1}{p(s)}\right)^{\Delta_t} \Delta s + \frac{1}{p(t)}\right) f\left(s, y_n^{\sigma}(s), y_n^{\tau}(s), y_n^{\xi}(s)\right) \Delta s\right| \\ &= \left|\frac{1}{p(t)} \int_t^{\infty} f\left(s, y_n^{\sigma}(s), y_n^{\tau}(s), y_n^{\xi}(s)\right) \Delta s\right| \\ &\leq \left|\frac{1}{p(t)} \int_t^{\infty} f\left(s, \beta, \beta, \frac{\beta}{2}\right) \Delta s\right| \\ &\leq \left|\frac{1}{p(t)} \int_t^{\infty} f\left(s, \alpha, \alpha, \frac{\alpha}{2}\right) \Delta s\right| \\ &\leq M. \end{aligned}$$

Therefore, the Ascoli–Arzela theorem along with a standard diagonalization argument yields a subsequence of $\{y_n(t)\}_{n=0}^{\infty}$ which converges uniformly on compact subintervals of $[T,\infty)_{\mathbb{T}}$ to a solution $y(t)$ of (1.3) satisfying $\beta/2 \le y(t) < \beta$, $t \ge T$. This proves the theorem. $\square$

To extend Theorems 2.1 and 2.2 to unbounded solutions, we introduce the class Φ of functions ϕ such that $\phi(u)$ denotes a continuous nondecreasing function of u satisfying $u\phi(u) > 0$, $u \neq 0$ with

$$\int_{\pm 1}^{\pm\infty} \frac{du}{\phi(u)} < \infty.$$

We will say that $f(t,u,v,w)$ satisfies condition (H) provided for some $\phi \in \Phi$ there exists $c \neq 0$ such that for all $t \ge T$

$$\inf_{|u|\ge c} \frac{f(t,u,\eta(t,T)u,\nu(t,T)u)}{\phi(u)} \ge k\left|f\left(t,c,\eta(t,T)c,\nu(t,T)c\right)\right|$$

for some positive constant k. We may now prove the following result.

Theorem 2.4. *Suppose $\phi \in \Phi$. Assume f satisfies conditions* (*A*), (*C*), *and* (*H*). *Then all solutions of*

$$(p(t)y^{\Delta})^{\Delta} + f\left(t, y^{\sigma}(t), y^{\tau}(t), y^{\xi}(t)\right) = 0 \tag{1.3}$$

are oscillatory in case

$$\left|\int^{\infty} kP(t,a) f\left(t,\alpha,\eta(t,T)\alpha,\nu(t,T)\alpha\right)\Delta t\right| = \infty \tag{2.7}$$

holds for all $\alpha \neq 0$ and for some sufficiently large $a \in \mathbb{T}$, where k is the constant appearing in condition (*H*).

Proof. Assume (2.7) holds for all $\alpha \neq 0$ and let u be a nonoscillatory solution of (1.3) with

$$u(t) > 0, \quad u(\tau(t)) > 0, \quad t \ge T.$$

As in the proof of Theorem 2.1, the BVP

$$[p(t)y^{\Delta}]^{\Delta} + F(t, y^{\sigma}(t)) = 0, \quad y(T) = u(T)$$

has a solution y with

$$u(T) \le y(t) \le u(t), \quad t \ge T,$$

where $F(t,u) = f(t,u,\eta(t,T)u,\nu(t,T)u)$. Also, as in the proof of Theorem 2.1, for $t > \tilde{t} > T$, we have

$$\int_{\tilde{t}}^{t} y^{\Delta}(s)\Delta s > \int_{\tilde{t}}^{t} P(r,\tilde{t})F(r,y^{\sigma}(r))\Delta r. \tag{2.8}$$

We next define the continuously differentiable real-valued function

$$G(u) := \int_{u_0}^{u} \frac{ds}{\phi(s)},$$

where $u_0 := y(T) > 0$. Observe that $G'(u) = 1/\phi(u)$. By the Pötzsche chain rule [6, Theorem 1.90],

$$(G \circ y)^{\Delta}(t) = \left(\int_0^1 \frac{dh}{\phi(y_h(t))}\right) y^{\Delta}(t) \geq \left(\int_0^1 \frac{dh}{\phi(y^{\sigma}(t))}\right) y^{\Delta}(t) = \frac{y^{\Delta}(t)}{\phi(y^{\sigma}(t))},$$

where $y_h(t) := y(t) + h\mu(t)y^{\Delta}(t) \leq y^{\sigma}(t)$. Consequently,

$$(G \circ y)^{\Delta}(t) \geq \frac{y^{\Delta}(t)}{\phi(y^{\sigma}(t))}.$$

Now multiplying (2.8) by $(\phi(y^{\sigma}(\tilde{s})))^{-1}$, we obtain

$$\begin{aligned}
\int_{\tilde{t}}^{t} \frac{y^{\Delta}(s)}{\phi(y^{\sigma}(s))}\Delta s &\geq \frac{1}{\phi(y^{\sigma}(s))}\int_{\tilde{t}}^{t} P(r,\tilde{t})F(r, y^{\sigma}(r))\Delta r \\
&\geq \int_{\tilde{t}}^{t} P(r,\tilde{t})\frac{F(r, y^{\sigma}(r))}{\phi(y^{\sigma}(r))}\Delta r \\
&\geq \int_{\tilde{t}}^{t} kP(r,\tilde{t})F(r,c)\Delta r
\end{aligned}$$

for sufficiently large $\tilde{t}$ (by condition (H)) where $c := u(T) > 0$. Furthermore, since $p(t)y^{\Delta}(t) > 0$, we have $y^{\Delta}(t) > 0$. It follows that $\lim_{t\to\infty} y(t) = L_2$ with $0 < L_2 < \infty$ and so

$$\lim_{t\to\infty} G(y(t)) = \lim_{t\to\infty} \int_{u_0}^{y(t)} \frac{du}{\phi(u)} = \int_{u_0}^{L_2} \frac{du}{\phi(u)} =: L < \infty.$$

Therefore as $t \to \infty$ we have

$$\int_{T}^{\infty} (G \circ y)^{\Delta}(s) = \lim_{t\to\infty} [G(y(t)) - G(y(T))] < \infty.$$

It follows that

$$\int_{\tilde{t}}^{t} (G \circ y)^{\Delta}(s)\Delta s \geq \int_{\tilde{t}}^{t} \frac{y^{\Delta}(s)}{\phi(y^{\sigma}(s))}\Delta s \geq \int_{\tilde{t}}^{t} kP(r,\tilde{t})F(r,c)\Delta r.$$

However the left side of the above is bounded as $t \to \infty$ whereas the right side is unbounded by assumption (2.7). This contradiction shows that all solutions of (1.3) are oscillatory. $\square$

3 Examples

We would like to illustrate some of the results above by means of examples. In the first example we need Lemma 1.1.

Example 3.1. Consider the linear functional dynamic equation

$$(p(t)y^{\Delta})^{\Delta} + q(t)y^{\sigma}(t) + r(t)y^{\tau}(t) + s(t)y^{\xi}(t) = 0, \tag{3.1}$$

where $p^{\Delta}(t) \geq 0$ on $[t_0, \infty)_{\mathbb{T}}$ and $p(t), q(t), r(t), s(t) > 0$, $t \geq t_0$. If we set

$$Q(t) := q(t) + r(t)\eta(t,T) + s(t)$$

for $t \geq T \geq t_0$, then (3.1) is oscillatory in case

$$(p(t)y^{\Delta})^{\Delta} + \lambda Q(t)y^{\sigma} = 0 \tag{3.2}$$

is oscillatory for some $0 < \lambda < 1$. To see this, suppose that u is a nonoscillatory solution of (3.1) with $u(t) > 0$, $u(\tau(t)) > 0$, $t \geq T$. Then, as in the work of Theorem 2.1, we have for all $t \geq T$

$$u(t) > 0, \quad u(\tau(t)) > 0, \quad u^{\Delta}(t) > 0, \quad [p(t)u^{\Delta}(t)]^{\Delta} < 0.$$

Then, by Lemma 1.1

$$\left(p(t)u^{\Delta}\right)^{\Delta} + \left(q(t) + r(t)\eta(t,T) + s(t)\right)u^{\sigma}(t) \leq 0, \quad t \geq T. \tag{3.3}$$

Then with $z(t) := \dfrac{p(t)u^{\Delta}(t)}{u(t)}$, using (3.3) we see that $z(t)$ satisfies the Riccati dynamic inequality (1.8) with $q(t) = Q(t)$. By Lemma 1.2, this means that the equation

$$\left(p(t)y^{\Delta}(t)\right)^{\Delta} + Q(t)y^{\sigma}(t) = 0$$

is nonoscillatory and so by the Sturm–Picone comparison theorem [16, Lemma 6], (3.2) is also nonoscillatory. This contradiction shows that (3.1) is oscillatory. If we apply a specific oscillation criterion and let $p(t) = 1$, we conclude that (3.1) is oscillatory if

$$\liminf\ t\int_t^{\infty} \left(q(u) + r(u)\eta(t,T) + s(u)\right)\Delta u > \frac{1}{4}$$

(see [15, Example 3.4]).

Example 3.2. Let $f(t,u,v,w) := q(t)u^{\gamma_1} + r(t)v^{\gamma_2}$, where $\gamma_1, \gamma_2 > 0$ are the quotients of odd positive integers. We assume also that $q(t), r(t) > 0$ for all large t and are rd-continuous. It is not difficult to show that both conditions (A) and (C) hold. By Theorem 2.1 all bounded solutions of

$$(p(t)y^{\Delta})^{\Delta} + q(t)(u^{\sigma}(t))^{\gamma_1} + r(t)(v^{\tau}(t))^{\gamma_2} = 0 \tag{3.4}$$

are oscillatory if

$$\int^{\infty} kP(t,a)\left[(q(t)\alpha)^{\gamma_1} + r(t)(\alpha k\eta(t,a))^{\gamma_2}\right]\Delta t = \infty \tag{3.5}$$

for all $\alpha \neq 0$ and for some $k \in (0,1)$. Now suppose $\gamma_1, \gamma_2 > 1$ and let $\phi(u) := u^{\gamma}$, where $1 < \gamma < \min\{\gamma_1, \gamma_2\}$. It is not difficult to show that $f(t,u,v) = p(t)u^{\gamma_1} + q(t)v^{\gamma_2}$ satisfies condition (H) with $k \leq |c|^{-\gamma}$. Therefore, from Theorem 2.4, we conclude that all solutions of (3.4) are oscillatory provided (3.5) holds.

As an illustration of the situation when f involves an advanced argument, we consider the following example.

Example 3.3. Suppose that

$$f(t,u,v,w) := \frac{q(t)u^{\gamma_1} + r(t)v^{\gamma_2}}{1 + s(t)w^2},$$

where $s(t) \geq 0$ is rd-continuous and $\gamma_1, \gamma_2, q(t), r(t) > 0$. From Theorem 2.1, we conclude that all bounded solutions of

$$(py^{\Delta})^{\Delta} + \frac{q(t)(y^{\sigma}(t))^{\gamma_1} + r(t)(y^{\tau}(t))^{\gamma_2}}{1 + s(t)(y^{\xi}(t))^2} = 0 \tag{3.6}$$

are oscillatory in case

$$\int^{\infty} P(t,a)\left(\frac{q(t)\alpha^{\gamma_1} + r(t)(\eta(t,a)\alpha)^{\gamma_2}}{1 + s(t)(\nu(t,a)M)^2}\right)\Delta t = \infty$$

for all $\alpha \neq 0$, where $M > 0$ is the upper bound of the oscillatory solutions of (3.6).

The results in the last two examples may be regarded as extensions of some oscillation criteria due to Atkinson [5].

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