Cauchy Functions and Taylor's Formula for Time Scales 𝕋

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Abstract In this paper we will be concerned with calculating the "Taylor monomials" that appear in the Taylor's formula for a function defined on a time scale. These Taylor monomials are very important for such Taylor series and are intimately related to Cauchy functions for certain dynamic equations. These Cauchy functions arise in variation of constants formulas and are also important when considering certain Green's functions. We will calculate several of these "Taylor monomials" for different time scales. In the last section we will give a short proof of Taylor's Theorem and give an interesting example.

Keywords Taylor monomials, time scales, Taylor's formula

AMS Subject Classification 39A10, 34B10

1 Introduction

The theory of time scales was introduced by Stefan Hilger in his 1988 PhD thesis [5] (supervised by Bernd Aulbach) in order to unify continuous and discrete analysis. The study of dynamical equations on time scales reveals discrepancies between continuous and discrete analysis, and often helps avoid proving results twice, once for differential equations and once for difference equations. The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale, which is an arbitrary closed subset of the reals. If we choose the time scale to be the set of the real numbers, the general result yields a result concerning an ordinary differential equation. If, on the other hand, we choose the time scale to be the set of integers, the general result yields a result for difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a more general result. Dynamic equations

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on time scales have a tremendous potential for applications. For example, it can model insect populations that are continuous while in season, die out in winter while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population. In Section 2 we will give several preliminary definitions that we will need in this paper. In Section 3 we will defined "Taylor monomials" and give several formulas for these Taylor monomials for various time scales. Finally in Section 4 we will give a short proof of Taylor's formula for a function on a time scale and give an interesting example. Argawal and Bohner [1] give a long proof of this Taylor's theorem but they use only basic results.

2   Preliminaries

First, we introduce some definitions. These definitions can be found in M. Bohner and A. Peterson [3] and R. P. Agarwal and M. Bohner [1].

Definition 1 *A* time scale $\mathbb{T}$ *is a nonempty closed subset of the reals.*

Definition 2 *Let* $\mathbb{T}$ *be a time scale. We define the* forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ *by*

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad for \quad t \in \mathbb{T},$$

while the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *is defined by*

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \quad for \quad t \in \mathbb{T}.$$

Here we put $\inf \emptyset = \sup \mathbb{T}$ *and* $\sup \emptyset = \inf \mathbb{T}$*, where* $\emptyset$ *denotes the empty set. For* $t \in \mathbb{T}$ *we say that* t *is* left-scattered *if* $\rho(t) < t$*, while if* $\sigma(t) > t$*, we say* t *is* right-scattered*. A point* $t \in \mathbb{T}$ *is* isolated *if it is both right-scattered and left-scattered at the same time. For* $t \in \mathbb{T}$ *we say* t *is* right-dense *if* $t < \sup \mathbb{T}$ *and* $\sigma(t) = t$*, while if* $t > \inf \mathbb{T}$ *and* $\rho(t) = t$*, we say* t *is* left-dense*. The* graininess function $\mu : \mathbb{T} \to [0, \infty)$ *is defined by*

$$\mu(t) := \sigma(t) - t.$$

If $\sup \mathbb{T} < \infty$ *and* $\sup \mathbb{T}$ *is* left-scattered*, we let* $\mathbb{T}^\kappa := \mathbb{T} \backslash \{\sup \mathbb{T}\}$*. Otherwise, we let* $\mathbb{T}^\kappa := \mathbb{T}$*.*

Definition 3 *Assume* $f : \mathbb{T} \to \mathbb{R}$ *is a function and let* $t \in \mathbb{T}^\kappa$*. Then we define* $f^\Delta(t)$ *to be the number (provided it exists) with the property that given any* $\varepsilon > 0$*, there is a neighborhood* U *of* t *such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \epsilon|\sigma(t) - s|$$

for all $s \in U$*. We call* $f^\Delta(t)$ *the delta (or Hilger) derivative of* f *at* t*. If* $\mathbb{T} = \mathbb{R}$*,* $f^\Delta = f'$*, whereas if* $\mathbb{T} = \mathbb{Z}$ *(the integers), then*

$$f^\Delta(t) = \Delta f(t) := f(t+1) - f(t),$$

that is, Δ *is the usual foward difference operator.*

Definition 4 *A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.*

If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is rd-continuous if and only if f is continuous. At the other extreme, if $\mathbb{T} = \mathbb{Z}$, then any function defined on $\mathbb{Z}$ is rd-continuous. It is known [3] that if f is rd-continuous, then there is a function F, called an antiderivative of f such that $F^\Delta(t) = f(t)$. In this case, we define

$$\int_a^b f(t)\Delta t = F(t)|_a^b.$$

If $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right hand side is the Riemann integral. If every point in $\mathbb{T}$ is isolated and $a < b$ are in $\mathbb{T}$, then we will use the formula (see [3])

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{\rho(b)} f(t)\mu(t).$$

3   Taylor Monomials

In this section we first define what we will call the *Taylor monomials* or *generalize polynomials* as defined originally by Agarwal and Bohner [1] and are also in the book by Bohner and Peterson [3]. They are called Taylor monomials because they are important, as we will see, in Taylor's formula for a function defined on a time scale. These Taylor monomials are also important because they are intimately related to Cauchy functions for certain dynamic equations which are important in variation of constants formulas. Since Green's functions are often given in terms of Cauchy functions, these Taylor monomials are important in the study of certain boundary value problems. The Taylor monomials $h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0$, are defined recursively as follows: The function h_0 is defined by

$$h_0(t, s) = 1, \quad \text{for all } s, t \in \mathbb{T},$$

and, given h_k for $k \in \mathbb{N}_0$, the function h_{k+1} is defined by

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s)\,\Delta\tau, \quad \text{for all } s, t \in \mathbb{T}.$$

If we let $h_k^\Delta(t, s)$ denote for each fixed $s \in \mathbb{T}$ the derivatives of $h_k(t, s)$ with respect to t, then

$$h_k^\Delta(t, s) = h_{k-1}(t, s), \quad \text{for } k \in \mathbb{N}, \quad t \in \mathbb{T}^\kappa.$$

The above definition obviously implies

$$h_1(t,s) = t - s, \quad \text{for all } s, t \in \mathbb{T}.$$

However, in general, finding h_k for $k \geq 2$ is very difficult. In this section we will give formulas for several of these important Taylor monomials for various time scales. In many of the applications it suffices to find formulas for $h_k(t,s)$ for just $t \geq s$. The first three examples that we give below are the formulas that are known in the literature and the other examples are new as far as we know.

Example 5 *If $\mathbb{T} = \mathbb{R}$, then*

$$h_k(t,s) = \frac{(t-s)^k}{k!}, \quad \text{for } t, s \in \mathbb{T}, \quad k \in \mathbb{N}_0.$$

Example 6 *Consider the time scale $\mathbb{T} = \mathbb{Z}$. The factorial function (see Kelley and Peterson [6]) $t^{\underline{k}}$ (read as t to the k falling), for $k \in \mathbb{N}_0$ is defined by $t^{\underline{0}} = 1$ and for $k \in \mathbb{N}$,*

$$t^{\underline{k}} = t(t-1)(t-2) \cdots (t-k+1).$$

In this case

$$h_k(t,s) = \frac{(t-s)^{\underline{k}}}{k!}, \quad \text{for } t, s \in \mathbb{T}, \ t \geq s, \quad k \in \mathbb{N}_0.$$

Example 7 (Agarwal and Bohner[1]) *Consider the time scale*

$$\mathbb{T} = \overline{q^{\mathbb{Z}}}, \quad \text{for some } q > 1.$$

This time scale is very important (see, e.g., Bézivin [2], G. Derfel, E. Romanenko, and A. Sharkovsky [4], Trijtzinsky [7], and Zhang [8]). In this case for $k \in \mathbb{N}_0$,

$$h_k(t,s) = \prod_{m=0}^{k-1} \frac{t - q^m s}{\sum_{j=0}^{m} q^j}, \quad \text{for all } s, t \in \mathbb{T}. \tag{1}$$

Example 8 *Consider the time scale with step size $h > 0$,*

$$\mathbb{T} = h\mathbb{Z} = \{0, \pm h, \pm 2h, \pm 3h, ...\}.$$

We claim that for $k \in \mathbb{N}_0$,

$$h_k(t,s) = \frac{\prod_{i=0}^{k-1}(t - ih - s)}{k!}, \quad \text{for all } s, t \in \mathbb{T}, \quad t \geq s \tag{2}$$

Evidently, for $k = 0$, the claim (2) holds (by convention $\prod_{i=0}^{-1} = 1$). Now we assume (2) holds with k replaced by some $m \in \mathbb{N}_0$. Then

$$
\begin{aligned}
h_{m+1}^{\Delta}(t,s) &= \frac{h_{m+1}(\sigma(t),s) - h_{m+1}(t,s)}{\mu(t)} \\
&= \frac{\prod_{i=0}^{m}(t+h-ih-s) - \prod_{i=0}^{m}(t-ih-s)}{(m+1)!h} \\
&= \frac{\prod_{i=-1}^{m-1}(t-ih-s) - \prod_{i=0}^{m}(t-ih-s)}{(m+1)!h} \\
&= \frac{\prod_{i=0}^{m-1}(t-ih-s)\left[(t+h-s) - (t-mh-s)\right]}{(m+1)!h} \\
&= \frac{\prod_{i=0}^{m-1}(t-ih-s)\left[t+h-s-t+mh+s\right]}{(m+1)!h} \\
&= \frac{\prod_{i=0}^{m-1}(t-ih-s)\,h\,[1+m]}{(m+1)!h} = \frac{\prod_{i=0}^{m-1}(t-ih-s)}{m!} = h_m(t,s)
\end{aligned}
$$

and since $h_{m+1}(s,s) = 0$ we get that (2) follows with k replaced by $m+1$. Hence by the principle of mathematical induction, (2) holds for all $k \in \mathbb{N}_0$.

Example 9 *Assume $\alpha_0 \in \mathbb{R}$ and $\alpha_k > 0$, $k \in \mathbb{N}$. Let $S := \{t = \sum_{k=0}^{n}\alpha_k,\ n \in \mathbb{N}_0\}$. If $\sum_{k=0}^{\infty}\alpha_k = \infty$, let $\mathbb{T}$ be the time scale $\mathbb{T} = S$, whereas if $L = \sum_{k=0}^{\infty}\alpha_k$ converges, let $\mathbb{T} = S \cup \{L\}$. We claim that for this time scale $\mathbb{T}$,*

$$
h_2(t,t_0) = \sum_{j=n_0+1}^{n-1}\sum_{k=n_0+1}^{j}\alpha_{j+1}\alpha_k, \quad \text{for } t \geq t_0, \tag{3}
$$

where $t = \sum_{k=0}^{n}\alpha_k$ and $t_0 = \sum_{k=0}^{n_0}\alpha_k$.
To see this let k_2 be defined for $t, t_0 \in \mathbb{T}$, $t \geq t_0$ by the right hand side of equation (3). Then by our convention on sums, $k_2(t_0,t_0) = 0$ and for $t \in \mathbb{T}$, $t \neq \sup\mathbb{T}$,

$$
\begin{aligned}
k_2^{\Delta}(t,t_0) &= \frac{k_2(\sigma(t),t_0) - k_2(t,t_0)}{\mu(t)} \\
&= \frac{\sum_{j=n_0+1}^{n}\sum_{k=n_0+1}^{j}\alpha_{j+1}\alpha_k - \sum_{j=n_0+1}^{n-1}\sum_{k=n_0+1}^{j}\alpha_{j+1}\alpha_k}{\alpha_{n+1}} \\
&= \frac{\sum_{k=n_0+1}^{n}\alpha_{n+1}\alpha_k}{\alpha_{n+1}} = \sum_{k=n_0+1}^{n}\alpha_k = t - t_0,
\end{aligned}
$$

which implies the desired result.

Example 10 *Consider the time scale*

$$
\mathbb{T} = \mathbb{N}_0^{\frac{1}{2}} = \{\sqrt{n} : n \in \mathbb{N}_0\}.
$$

Note that

$$\sigma(t) = \sqrt{n+1}, \quad \mu(t) = \sqrt{n+1} - \sqrt{n},$$

where $t = \sqrt{n}$, $n \in \mathbb{N}_0$. For this time scale we will just find $h_2(t,0)$.
Consider

$$
\begin{aligned}
h_2(t,0) &= \int_0^\tau 1 \Delta\tau \\
&= 0\mu(0) + \sqrt{1}\mu\left(\sqrt{1}\right) + \cdots + \mu\sqrt{n-1}\left(\sqrt{n-1}\right) \\
&= \sqrt{0}\left(\sqrt{1} - \sqrt{0}\right) + \sqrt{1}\left(\sqrt{2} - \sqrt{1}\right) + \ldots + \sqrt{n-1}\left(\sqrt{n} - \sqrt{n-1}\right) \\
&= \sum_{k=0}^{n-1} \sqrt{k}\left(\sqrt{k+1} - \sqrt{k}\right) = \sum_{k=0}^{n-1}\left[\left(\sqrt{k(k+1)}\right) - k^{\underline{1}}\right] \\
&= p(t) - \left.\frac{k^{\underline{2}}}{2}\right|_0^n = p(t) - \frac{n(n-1)}{2} = p(t) - \frac{t^2\left(t^2-1\right)}{2},
\end{aligned}
$$

where $p(t) = \sum_{s \in [0,t)} s\sigma(s)$.

Example 11 *We consider the time scale*

$$\mathbb{T} = \mathbb{N}_0^2 = \{n^2 : n \in \mathbb{N}_0\}.$$

Note that

$$\sigma(t) = (n+1)^2, \quad \mu(t) = 2n+1,$$

where $t = n^2, n \in \mathbb{N}_0$.
Consider

$$
\begin{aligned}
h_2(t,0) &= \int_0^t \tau\Delta\tau = 0^2\mu\left(0^2\right) + 1^2\mu\left(1^2\right) + \cdots + (n-1)^2\mu\left((n-1)^2\right) \\
&= 1^2(2(1)+1) + 2^2(2(2)+1) + \cdots + (n-1)^2(2(n-1)+1) \\
&= \sum_{k=1}^{n-1} k^2(2k+1) = \sum_{k=1}^{n-1}\left(2k^3 + k^2\right) = \sum_{k=1}^{n-1}\left(2k^{\underline{3}} + 7k^{\underline{2}} + 3k^{\underline{1}}\right) \\
&= \left.\frac{k^{\underline{4}}}{2} + \frac{7k^{\underline{3}}}{3} + \frac{3k^{\underline{2}}}{2}\right|_1^n = \frac{n^{\underline{4}}}{2} + \frac{7n^{\underline{3}}}{3} + \frac{3n^{\underline{2}}}{2} \\
&= \frac{n(n-1)(n-2)(n-3)}{2} + \frac{7n(n-1)(n-2)}{3} + \frac{3n(n-1)}{2} \\
&= \frac{n(n-1)\left(3n^2 - n - 1\right)}{6} = \frac{\sqrt{t}\left(\sqrt{t}-1\right)\left(3t - \sqrt{t} - 1\right)}{6}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
h_3\left(t,0\right) \;&=\; \int_0^t h_2(\tau,0)\Delta\tau \;=\; \sum_{k=0}^{n-1} h_2\left(k^2,0\right)\mu\left(k^2\right) \\
&=\; \sum_{k=0}^{n-1} \frac{k\left(k-1\right)\left(3k^2-k-1\right)}{6}\left(2k+1\right) \\
&=\; \sum_{k=0}^{n-1}\left(k^5-\frac{5k^4}{6}-\frac{2k^3}{3}+\frac{k^2}{3}+\frac{k}{6}\right) \\
&=\; \sum_{k=0}^{n-1}\left(k^{\underline{5}}+\frac{55k^{\underline{4}}}{6}+\frac{58k^{\underline{3}}}{3}+\frac{15k^{\underline{2}}}{2}\right) \\
&=\; \left(\frac{k^{\underline{6}}}{6}+\frac{11k^{\underline{5}}}{6}+\frac{29k^{\underline{4}}}{6}+\frac{5k^{\underline{3}}}{3}\right)\Bigg|_0^n \\
&=\; \frac{n^{\underline{6}}}{6}+\frac{11n^{\underline{5}}}{6}+\frac{29n^{\underline{4}}}{6}+\frac{5n^{\underline{3}}}{3} \\
&=\; \frac{n\left(n-1\right)\left(n-2\right)\left(n^3-n^2-n-5\right)}{6} \\
&=\; \frac{\sqrt{t}\left(\sqrt{t}-1\right)\left(\sqrt{t}-2\right)\left(t^{\frac{3}{2}}-t-t^{\frac{1}{2}}-5\right)}{6}.
\end{aligned}
$$

Now we will examine $h_2(t,s))$ for $t \geq s$. Let $m=\sqrt{s}$.

$$
\begin{aligned}
&h_2(t,s) \\
=\;& \sum_{k=m}^{n-1}\left(k^2-m^2\right)\left(2k+1\right) \\
=\;& \frac{n^{\underline{4}}}{2}+\frac{7n^{\underline{3}}}{3}+\frac{3n^{\underline{2}}}{2}-\frac{m^{\underline{4}}}{2}-\frac{7m^{\underline{3}}}{3}-\frac{3m^{\underline{2}}}{2}-m^2\left(n^{\underline{2}}+n\right)+m^2\left(m^{\underline{2}}+m\right) \\
=\;& \frac{(n-m)^{\underline{2}}}{3!}\times \\
& \times\left\{3(n-m-2)^{\underline{2}}+(12m+14)(n-m-2)+(12m^2+24m+9)\right\} \\
=\;& \frac{\left(\sqrt{t}-\sqrt{s}\right)^{\underline{2}}}{3!}\times \\
& \times\left\{3(\sqrt{t}-\sqrt{s}-2)^{\underline{2}}+(12\sqrt{s}+14)(\sqrt{t}-\sqrt{s}-2)+(12s+24\sqrt{s}+9)\right\}
\end{aligned}
$$

4   Taylor's Theorem

In Agarwal and Bohner [1] a elementary but lengthy proof of Taylor's Theorem for a function on a time scale is proved. In this section we will give a short proof of this Taylor's Theorem but it will depend on knowing more results.

Theorem 12 *(Taylor's Formula). Assume $f \in C^{n+1}_{rd}(\mathbb{T})$ and $s \in \mathbb{T}$. Then*

$$f(t) = \sum_{k=0}^{n} f^{\Delta^k}(s)\, h_k(t,s) + \int_s^t h_n(t, \sigma(\tau))\, f^{\Delta^{n+1}}(\tau)\, \Delta\tau. \qquad (4)$$

Proof. Let g be defined by

$$g(t) := f^{\Delta^{n+1}}(t).$$

Then f is the unique solution of the IVP

$$x^{\Delta^{n+1}} = g(t), \quad x^{\Delta^i}(s) = f^{\Delta^i}(s), \quad 0 \le i \le n.$$

Let

$$u(t) := \sum_{k=0}^{n} f^{\Delta^k}(s)\, h_n(t,s)$$

and

$$v(t) := \int_s^t h_n(t, \sigma(\tau))g(\tau)\, \Delta\tau.$$

Then u solves the IVP

$$u^{\Delta^{n+1}} = 0, \quad u^{\Delta^i}(s) = f^{\Delta^i}(s), \quad 0 \le i \le n.$$

It suffices to show v solves the IVP

$$v^{\Delta^{n+1}} = g(t), \quad v^{\Delta^i}(s) = 0, \quad 0 \le i \le n.$$

Clearly $v(s) = 0$. Also

$$
\begin{aligned}
v^{\Delta}(t) &= \int_s^t h_n^{\Delta}(t, \sigma(\tau))\, g(\tau)\Delta\tau + h_n(\sigma(t), \sigma(t))\, g(t) \\
&= \int_s^t h_n^{\Delta}(t, \sigma(\tau))\, g(\tau)\Delta\tau.
\end{aligned}
$$

Note $v^{\Delta}(s) = 0$ and

$$
\begin{aligned}
v^{\Delta^2}(t) &= \int_s^t h_n^{\Delta^2}(t, \sigma(\tau))\, g(\tau)\Delta\tau + h_n(\sigma(t), \sigma(t))\, g(t) \\
&= \int_s^t h_n^{\Delta^2}(t, \sigma(\tau))\, g(\tau)\Delta\tau.
\end{aligned}
$$

Note $v^{\Delta^2}(s) = 0$. Proceeding in this manner we obtain by mathematical induction that

$$v^{\Delta^i}(t) = \int_s^t h_n^{\Delta^i}(t, \sigma(\tau))\, g(\tau)\Delta\tau$$

for $0 \le i \le n$ and $v^{\Delta^i}(s) = 0$, $0 \le i \le n$. Finally,

$$
\begin{aligned}
v^{\Delta^{n+1}}(t) &= \int_s^t h_n^{\Delta^{n+1}}(t, \sigma(\tau)) g(\tau) \Delta \tau + h_n(\sigma(t), \sigma(t)) g(t) \\
&= g(t),
\end{aligned}
$$

and this proof is complete. ∎

The following example was motivated by an example shown to the authors by Douglas Anderson. This is an example of a function on the time scale $\mathbb{T} = \mathbb{Z}$, where the Taylor series of a function about $t = 0$ converges to the function for $t \ge 0$ but diverges for $t < 0$.

Example 13 *For* $\mathbb{T} = \mathbb{Z}$*, consider* $f(t) = e_1(t, 0) = 2^t$ *for* $t \in \mathbb{Z}$*. If we expand* f *about* 0*, then Taylor's formula* (4) *for* f *is given by*

$$
f(t) = 2^t = P_n(t) + E_n(t),
$$

where the Taylor polynomial P_n *is given by*

$$
P_n(t) = \sum_{k=0}^n \frac{t^{\underline{k}}}{k!}
$$

and the error term E_n *is given by*

$$
E_n(t) = \begin{cases} \sum_{\tau=1}^{t-1} (t - \tau - 1)^{\underline{n}} 2^\tau & if \quad t > 0 \\ 0 & if \quad t = 0 \\ -\sum_{\tau=t+1}^0 (t - \tau - 1)^{\underline{n}} 2^\tau & if \quad t < 0. \end{cases}
$$

Then $P_n(-1) = \frac{1 - (-1)^{n+1}}{2}$ *and* $E_n(-1) = \frac{(-1)^{n+1}}{2}$*, so that the Taylor polynomial will not converge to* f *at* -1 *as* $n \to \infty$*. Note, however, that*

$$
f(t) = 2^t = \sum_{k=0}^\infty \frac{t^{\underline{k}}}{k!} = P_t(t) = \sum_{k=0}^t \binom{t}{k}
$$

for any nonnegative integer t*. It is not difficult to show that the Taylor's series for* $f(t) = 2^t$ *with respect to the time scale* $\mathbb{T} = \mathbb{Z}$ *diverges for any integer* $t < 0$*.*

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