Math 1331 Final Exam, Spring 2015 (v. A)

In section A), you must show your work if you are to receive credit. Present the problems in your blue book in the order that they occur on the exam: problem 1 first, problem 2 second, etc. You may use your graphing calculator, but no other types of electronic devices. Books, notes, formula sheets are not allowed. The number of points assigned per problem is indicated in the parenthesis.

A). Show-work Problems

1). The demand for x units of a certain item is
$p = D(x) = 100 + 50/(\ln x)$, where p is measured in dollars and $x > 1$.

a). **(5 p)** Show that $p = D(x)$ is a decreasing function of x.

b). **(5 p)** Find the limit of this function as the value x goes to infinity.

c). **(5 p)** Find the *revenue*, and then the *marginal revenue* (that is, the derivative of the revenue).

2). A company manufactures television sets. The quantity x of these sets (in thousands) demanded each week is related to the wholesale unit price p by the following equation: $p = D(x) = 100 - 25x$. The weekly total cost incurred for producing x sets is given by $C(x) = 50x + 100$.

a.) **(5 p)** Find the average cost function.
b). **(5 p)** Find the revenue function $R(x)$.
c). **(5 p)** Find the marginal revenue function.
d). **(5 p)** Find the profit function $P(x)$.
e). **(5 p)** Find the marginal profit function.

3). Suppose that the total profit (expressed in hundreds of dollars) from selling x items is given by:

$$P(x) = x^2 - 2x + 5$$

a). **(5 p)** Find and interpret the instantaneous rate of change of profit with respect to the number of items produced when $x = 50$ (this number is called the marginal profit at $x = 50$).

b). **(5 p)** Find the positive values x for which the profit is minimized or maximized, if such values exist. Does the profit attain its minimum or maximum?

4). Study the behavior of the profit function $P(x) = x^3 - 3x^2 + 3x$ for a non-negative value x.

a). **(5 p)** Indicate the exact intervals where function is decreasing or increasing, respectively.

b). **(5 p)** Find the absolute extrema of the function $P(x)$ over the interval [0, 5].

5). (5 p) The marginal cost function for producing a certain type of item is given by $C'(x) = \frac{2x}{x^2+2}$. The fixed cost is \$10. What is the cost function?

6). (5 p) Let $q = f(p)$, where q represents the demand at price p. The elasticity of demand is defined as $E = \left(-\frac{p}{q}\right) \frac{dq}{dp}$. Let the demand be given by the function $q = 50 - \frac{p}{4}$ where p represents the price per item. Find the expression of function E, and the value q for which the total revenue is maximized.

7). (5 p) Compute $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ for the function $f(x) = \frac{1}{x}$. What is the meaning of this limit?

8). (5 p) Compute the derivative of the function $f(x) = e^{7\,x^2+1}$ and write the equation of the tangent line to the graph of $f(x)$, at the point (0, e).

9). Suppose the supply function for oil (in \$) is given by $S(q) = q^2 + 10\,q$ and the

demand function is given (in \$) by $D(q) = 200 + 10\,q - q^2$.

(5 p) a). Find the point in which supply and demand are in equilibrium.

(5 p) b). Find the consumer's surplus and the producer's surplus.

B). Multiple Choice Problems

10). (5 p) Given a revenue function R with derivative $R'(x) = \frac{1}{x}$, for any value x greater than or equal to 1, such that $R(1) = 0$, what is the value of the revenue at $x = 2$?

a). 0; b). 1; c). -1; d). ln 2; e). none of the previous

11). (5 p) Find the relative extrema (maxima or minima) of the profit function $P(x) = x^3 - 3\,x^2 + 3\,x$.

The answer is: a). 0; b). 1; c). $0 \; and \; 1$; d). ln 2; e). none of the previous

12). (5 p) Given $R\,(x) = (x + 5)(\ln x)$, compute $R'(1)$. Answer is:

a). 0; b). 1; c). 6; d). none of the previous

13). (5 p) For the function $f(x) = 3x(x^2 + 1)^5$, find $f'(0)$.

a). 0 ; b). 1; c). 3 d). none of the previous

14). (5 p) Find the marginal profit function $P'(x)$ for the profit function $P(x) = e^{-4x}$. Evaluate the *limit of the marginal profit function* as the production level x becomes infinitely large. The limit represents:

a). e ; b). -4; c). 0; d). does not exist

15). (5 p) If $x^2 + y^2 = 1$, use implicit differentiation to find dy/dx in terms of x and y.

a). x/y; b). y/x; c). $-y/x$; d). $-x/y$; e). none of the previous

16). (5 p) Compute the *third order derivative* of the function $f(x) = \ln x$. That function is:

a). $-1/x^2$; b). $1/x$; c) $2/x^3$; d). none of the previous

SECTION A: Show-work Problems

1.

(a) $p = D(x)$ is a decreasing function of x, since $\ln x$, $x > 1$ is increasing.
(Another argument would be

$$D'(x) = -50 \cdot \frac{1}{(\ln x)^2} \cdot \frac{1}{x} < 0,$$

for all values in the domain.)

(b)

$$\lim_{x \to \infty} D(x) = 100.$$

(c)

$$R(x) = x \cdot p = 100x + \frac{50x}{\ln x}$$

$$R'(x) = 100 + \frac{50 \cdot \ln x - 50x\frac{1}{x}}{(\ln x)^2} = 100 + \frac{50}{\ln x} - \frac{50}{(\ln x)^2}.$$

2.

(a)

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{50x + 100}{x} = 50 + \frac{100}{x}.$$

(b) $R(x) = p \cdot x = (100 - 25x) \cdot x = 100x - 25x^2$.

(c) $R'(x) = 100 - 50x$.

(d)

$$P(x) = R(x) - C(x) = (100x - 25x^2) - (50x + 100),$$

$$P(x) = -25x^2 + 50x - 100 = -25(x^2 - 2x + 4).$$

(e) $P'(x) = -25(2x - 2) = -50x + 50$.

3.

(a) $P'(x) = 2x - 2 \Rightarrow P'(50) = 100 - 2 = 98$.

(b) $P(x)$ has no maximum. $P(x)$ attains its minimum at 1 (either by the theory of quadratic function, $x_0 = -\frac{(-2)}{2.1} = 1$ or by setting $P'(x) = 0$ and P concave up $\Rightarrow x_0 = 1$ gives the critical value).

$P(1) = 4 \rightarrow$ minimum, attained.

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4. $P(x) = x^3 - 3x^2 + 3x, \quad x \geq 0.$

(a) $P'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 \geq 0$ for all x values $\Rightarrow P$ is increasing for all x.

(b)

$$P(0) = 0$$
$$P(5) = 5^3 - 3 \cdot 5^2 + 3 \cdot 5 = 125 - 75 + 15 = 65$$
$$\Rightarrow P(5) = 65 \quad \text{represent the absolute maximum on} \quad [0,5].$$

5.

$$C'(x) = \frac{2x}{x^2+2} \Rightarrow C(x) = \ln(x^2+2) + K$$
$$C(0) = \ln 2 + K = 10 \Rightarrow K = 10 - \ln 2.$$

The cost function is $C(x) = \ln(x^2+2) + 10 - \ln 2$.
It can also be expressed as $C(x) = \ln\dfrac{(x^2+2)}{2} + 10$.

6.

$$q = 50 - \frac{p}{4} \Rightarrow E = -\frac{p}{q} \cdot \frac{dq}{dp} = -\frac{p}{50 - \frac{p}{4}} \cdot \Big(-\frac{1}{4}\Big) = \frac{p}{200-p}.$$

From theory, we know that the revenue is maximized when $E = 1$.

$$E = 1 \Leftrightarrow 200 - p = p \Leftrightarrow p = 100 \Leftrightarrow q = 25.$$

7.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0} \frac{x-x-h}{x(x+h)h} = -\frac{1}{x^2},$$

which represent derivative at value x.

8.

$$f(x) = e^{7x^2+1} \Rightarrow f'(x) = 14xe^{7x^2+1} \Rightarrow f'(0) = 0.$$

Tangent line at $(0,e)$ is: $y - e = 0 \Leftrightarrow y = e$.

9.

$$S(q) = q^2 + 10q,$$
$$D(q) = 200 + 10q - q^2.$$

(a)

$$S(q) = D(q) \Rightarrow q^2 + 10q = 200 + 10q - q^2 \Leftrightarrow q^2 = 100 \Leftrightarrow q_0 = 10.$$

(b)

$$q_0 = 10 \Rightarrow p_0 = q_0^2 + 10q_0 = 100 + 100 = 200.$$
$$\text{CS=Consumer's surplus} = \int_0^{q_0} [D(q) - p_0]dq.$$
$$\text{PS=Producer's surplus} = \int_0^{q_0} [p_0 - S(q)]dq.$$

$$\text{CS} = \int_0^{10} [200 + 10q - q^2 - 200] dq = \left[5q^2 - \frac{q^3}{3}\right]_{q=0}^{q=10} = 500 - \frac{1000}{3} \simeq 166.67\$.$$

$$\text{PS} = \int_0^{10} [200 - q^2 - 10q] dq = \left[200q - \frac{q^3}{3} - 5q^2\right]_0^{10} = 2000 - \frac{1000}{3} - 500 = 1500 - \frac{1000}{3} \simeq 1166.67\$.$$

SECTION B: Multiple Choice Problems

10.

$$R'(x) = \frac{1}{x}, \quad R(1) = 0 \Rightarrow R(x) = \ln x \Rightarrow R(2) = \ln 2.$$

Answer: $d) \ln 2$.

11.

$$P(x) = x^3 - 3x^2 + 3x; \quad P'(x) = 3(x-1)^2 \geq 0$$

Note P is increasing, taking all real values between $-\infty$ and ∞. There are no local minima and maxima only an inflection point at $x = 1$.
Answer: $e)$ None of previous.

12.

$$R'(x) = 1 \cdot \ln x + \frac{x+5}{x} \Rightarrow R'(1) = \ln 1 + 6 = 6.$$

Answer: $c) 6$

13.

$$f'(x) = 3 \cdot (x^2+1)^5 + 3x \cdot 5(x^2+1)^4 \cdot 2x \Rightarrow f'(0) = 3.$$

Answer: $c) 3$

14.

$$P(x) = e^{-4x} \Rightarrow P'(x) = -4e^{-4x} \Rightarrow \lim_{x \to \infty} P'(x) = 0.$$

Answer: $c) 0$

15.

$$x^2 + y^2(x) = 1 \Rightarrow 2x + 2y \cdot y' = 0 \Rightarrow y'(x) = -\frac{x}{y}.$$

Answer: $d) -\frac{x}{y}$

16.

$$f'(x) = \frac{1}{x}, f''(x) = -\frac{1}{x^2} = -x^{-2}, f'''(x) = \frac{2}{x^3}.$$

Answer: $c) \frac{2}{x^3}$