1331 Final Exam, Fall 2016

You must show your work, and the work you show must yield the answer you obtain, if you are to receive credit. Present the problems in your blue book in the order that they occur on the exam: problem 1 first, problem 2 second, etc. Allow at least one full page for each problem. Note that there is a 20 point bonus question on this exam (#11).

Derivative formulas

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx} \qquad \frac{d}{dx}e^u = e^u\frac{du}{dx} \qquad \frac{d}{dx}\ln(u) = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx} \qquad \frac{d}{dx}\frac{u}{v} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Integral formulas

$$\int u^n\frac{du}{dx}dx = \frac{u^{n+1}}{n+1} + C \quad for \quad n \neq 1 \qquad \int \frac{1}{u}\frac{du}{dx}dx = \ln|u| + C \qquad \int e^u\frac{du}{dx}dx = e^u + C$$

Continuous Income Stream formulas

Present value $= \int_0^k f(t)e^{-rt}dt$ Future value $= e^{rk}\int_0^k f(t)e^{-rt}dt$

1. (20p) **Basic Knowledge and Skills (Derivatives and Integrals)**. Please compute the following derivatives and integrals (each problem is worth 2 points):

   a. $\frac{d}{dx}\frac{1}{x^2+2x}$ b. $\frac{d}{dt}(t^3+1)e^{2t}$ c. $\frac{d}{dx}\frac{2x^3+x}{2x+1}$

   d. $\frac{d}{dp}\sqrt{p^{-2}+1}$ e. $\frac{d}{dy}\ln(2y^2+1)$ f. $\int \frac{3}{x^2} - x\,dx$

   g. $\int (x^2+3)^2 x\,dx$ h. $\int (s^2+1)^2\,ds$ i. $\int -3e^{4y}\,dy$

   j. $\int \frac{3t}{t^2+1}\,dt$

2. (20p) **Cost-Benefit.** Suppose the cost C of removing p percent impurities of particulate pollution from the smoke-stacks of an industrial plant is given by

$$C(p) = \frac{700{,}000}{100-p} - 7000.$$

a. (4p) What is the (complete) domain of the function $C(p)$? What are the values of p that are allowed in the context of this particular problem?
b. (4p) How much does it cost to remove 80% of the particulate pollution form the smoke-stakes?
c. (4p) Find $\lim_{p \to 80} C(p)$.
d. (4p) Find $\lim_{p \to 100^-} C(p)$.
e. (4p) Is complete purity possible? Explain.

3. (20p) **Cost and Average Cost.** Suppose that the total cost function, in dollars, for the production of x units of a product is given by

$$C(x) = 5000 + 50x + 0.1x^2.$$

a. (4p) What is the fixed cost?
b. (4p) What is the average cost function $\bar{C}(x)$?
c. (4p) Find the instantaneous rate of change of average cost with respect to the number of units produced at any level of production.
d. (4p) Find the level at which this rate of change equals 0.
e. (4p) At the value found in (d) find the instantaneous rate of change of the cost function and the value of the average cost. What do you observe?

4. (20p) **Revenue.** A travel agency will plan a group tour for groups of size 30 or larger. If the group contains exactly 30 people, the cost is \$400 per person. Suppose each person's cost is reduced by \$10 for each additional person above 30.

a. (4p) What is the revenue from a group of 30 people?
b. (4p) What is the revenue from a group of 35 people?
c. (4p) Let x represent the number of additional people above 30. Find the revenue function $R(x)$.
d. (4p) Find the marginal revenue function.
e. (4p) What is the size of a group that will produce the maximum revenue, and what is that maximum revenue?

5. (20p) **Diminishing Return**. Suppose that the total number of units produced by a worker in t hours into an 8-hour shift can be modeled by the production function

$$P(t) = 27t + 12t^2 - t^3.$$

   a. (4p) Sketch a graph of $P(t)$ on the interval $[\,0\,,8\,]$. (Use your graphing calculator with then window $0 \le x \le 8$ and $0 \le y \le 500$.)
   b. (4p) What is the (instantaneous) rate of change at exactly 4 hours into the shift?
   c. (4p) What is the (instantaneous) rate of change at exactly 7 hours and 30 minutes into the shift?
   d. (4p) Find the point of diminishing returns. (The point where the rate $P'(t)$ is maximal, ie were the rate stops increasing and starts decreasing.)
   e. (4p) Is the answer in (d) consistent with your graph in (a)? Explain.

6. (20p) **Inventory Model.** A company needs 450,000 items per year. Production costs are $500 to prepare for a production run and $10 for each item produced. Inventory costs are $2 per item per year. (Assume the demand is constant throughout the year.)
   a. (4p) What would be the total cost to the company if all 450,000 items were purchased and stored at the beginning of the year? (Assume that the demand is constant throughout the year, and that the average number in storage throughout the year is $\frac{450000}{2}$.)
   b. (4p) What would be the total cost to the company if it placed 5 equally spaced orders throughout the year? (Note that each order would be for $\frac{450000}{5} = 90000$.)
   c. (4p) What would be the total cost function if the company placed n equally spaced orders throughout the year?
   d. (4p) What is the number of orders per year that should be placed in order to minimize the total cost.
   e. (4p) What is the minimal annual cost?

7. (20p) **Cost and Marginal Cost.** Suppose the total cost (in dollars) is given by

$$C(x) = 2000 + 200\ln(2x + 2)$$

where x is the number of units produced.

a. (4p) Find the fixed cost?
b. (4p) Find the total cost of making 200 units.
c. (4p) Find the marginal cost function.
d. (4p) Find the marginal cost when 200 units are produced.
e. (4p) Find the actual cost of the 201st unit.

8. (20p) **Marginal Profit and Profit.** A firm's marginal cost for a product is $\overline{MC} = 2x + 50$, and its marginal revenue is $\overline{MR} = 200 - 4x$, and the cost of production of 10 items is $700.

a. (4p) Find the optimal level of production. (The value of x that maximizes profit.)
b. (4p) Find the Cost function.
c. (4p) Find the Revenue function.
d. (4p) Find the Profit function.
e. (4p) Find the maximum profit.

9. (20p) **Investing.** When the interest on an investment is compounded continuously, the investment grows at a rate that is proportional to the amount in the account. If P is the amount invested in the account then

$$\frac{dP}{dt} = kP,$$

where t is in years and k is the proportionally constant.

a. (4p) Solve this differential equation for the value of P as function of t. (In general)
b. (4p) Suppose that $200,000 is invested in the account (at t=0). What is the specific value of P as function of t? (The proportionally constant is still k.)
c. (4p) Suppose the amount in the account after 15 years is $400,000. What is the proportionally constant, k?
d. (4p) How much is in the account after 20 years?
e. (4p) What is the interest rate on this investment?

10. (20p) **Consumer and Producer Surplus.** Suppose the demand function for a product is given by $D(x) = -\frac{1}{2}x + 10$ and the supply function is given by $S(x) = \frac{1}{5}x^2$ where x is the number of units.

a. (4p) Sketch the graphs of $D(x)$ and $S(x)$. Use your calculator with the window $0 \leq x \leq 20 \; and \; 0 \leq y \leq 10$.

$b.$ (4p) Compute the equilibrium point and plot it on the graph

$c.$ (4p) Identify and shade in the regions that represent the consumer's and the producer's surplus.

d. (4p) Compute the producer's surplus.

e. (4p) Compute the consumers surplus.

11. (20) **Continuous Income Flow.** Suppose a continuous income stream has an annual rate of flow at time t is given by

$$f(t) = 10e^{-0.4(t+3)}$$

in thousands of dollars. Also suppose the money is worth 8% compounded continuously.

a. (4p) Sketch the graph of $f(t)$. Use your calculator with window $0 \leq x \leq 10$ and $0 \leq y \leq 5$.

b. (4p) Shade in the region that represents the total income over the first 5 years.

c. (4p) What will be the total income from this income stream over the first 5 years?

d. (4p) What is the present value of the income stream over the next 5 years?

e. (4p) What is the future value of the income stream over the next 5 years?