(1) Evaluate the following limits:

(i) \[ \lim_{x \to 1} \frac{x}{x^2 + 1} \]

(ii) \[ \lim_{x \to \infty} \frac{x}{\sqrt{x^2 - x + 1}} \]

(iii) \[ \lim_{x \to \infty} x^{-5} \ln(x) \]

(iv) \[ \lim_{x \to \infty} x^{1/x} \]

Solution

(i)

\[ \lim_{x \to 1} \frac{x}{x^2 + 1} = \frac{1}{1 + 1} = \frac{1}{2} \]

(ii)

\[ \lim_{x \to \infty} \frac{x}{\sqrt{x^2 - x + 1}} = \lim_{x \to \infty} \frac{x}{\sqrt{x^2(1 - \frac{1}{x}) + 1}} \]

\[ = \lim_{x \to \infty} \frac{x}{x \left( \sqrt{1 - \frac{1}{x}} + 1 \right)} \]

\[ = \lim_{x \to \infty} \frac{1}{\sqrt{1 - \frac{1}{x} + 1}} \]

\[ = \frac{1}{2} \]

(iii)

\[ \lim_{x \to \infty} x^{-5} \ln(x) = \lim_{x \to \infty} \frac{\ln(x)}{x^5} \]

\[ \equiv \lim_{x \to \infty} \frac{d}{dx} \ln(x) \]

\[ = \lim_{x \to \infty} \frac{1}{5x^4} \]

\[ = \lim_{x \to \infty} \frac{1}{5x^5} \]

\[ = 0 \]

(iv) Let \( L = \lim_{x \to \infty} x^{1/x} \). Then

\[ \ln(L) = \lim_{x \to \infty} \frac{\ln(x)}{x} \]

\[ \equiv \lim_{x \to \infty} \frac{d}{dx} \ln(x) \]

\[ = \lim_{x \to \infty} \frac{1}{x} = 0 \]

\[ = 0 \]

and so \( L = e^0 = 1 \).
(2) For each function below find \( \frac{dy}{dx} \):

(i) \( y = \left( \frac{x^2 + 5}{x^2 - 5} \right)^3 \)

(ii) \( y = \tan^{-1}(3x) \)

(iii) \( y = \ln(\sin^2(x)) \)

(iv) \( y = 8x \sec^4(x^3) \)

Solution

(i)

\[
\frac{dy}{dx} = 3 \left( \frac{x^2 + 5}{x^2 - 5} \right)^2 \frac{(x^2 - 5)(2x) - (x^2 + 5)(2x)}{(x^2 - 5)^2}
\]

(ii)

\[
\frac{dy}{dx} = \frac{3}{1 + (3x)^2} = \frac{3}{1 + 9x^2}
\]

(iii)

\[
\frac{dy}{dx} = \frac{1}{\sin^2 x} (2 \sin x \cos x) = 2 \cot x
\]

(iv)

\[
\frac{dy}{dx} = 8 \cdot \sec^4(x^3) + 8x \cdot 4 \sec^3(x^3) \sec(x^3) \tan(x^3) 3x^2
\]

(3) Using implicit differentiation, find \( \frac{dy}{dx} \) when \( e^{xy} = 3y^2 - 2 \ln(x) \).

Solution

\[
\frac{d}{dx} [e^{xy}] = \frac{d}{dx} [3y^2 - 2 \ln(x)]
\]

\[
e^{xy}(xy' + y) = 6yy' - \frac{2}{x}
\]

\[
y' = \frac{-\frac{2}{x} - e^{xy}y}{e^{xy}x - 6y}
\]

(4) Find the equation of the tangent line to the graph of the function \( f(x) = \tan \left( \frac{x}{4} \right) \) at \( x = \pi \).

Solution First \( f(\pi) = 1 \). Then \( f'(x) = \frac{1}{4} \sec^2 \left( \frac{x}{4} \right) \Rightarrow f'(\pi) = \frac{1}{2} \). Using point-slope form, the equation of the tangent line at \( x = \pi \) is

\[
y - 1 = \frac{1}{2}(x - \pi)
\]

\[
y = \frac{1}{2}x - \frac{\pi}{2} + 1.
\]
(5) A 13 ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 6 ft/sec. How fast is the top of the ladder sliding down the wall at this point? Include units.

Solution

\[ x = 12, \quad \frac{dx}{dt} = 6 \text{ ft/sec} \]

\[ 12^2 + y^2 = 13^2 \]
\[ y^2 = 169 - 144 \]
\[ y^2 = 25 \]
\[ y = 5 \]

\[ x^2 + y^2 = 13^2 \]
\[ 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0 \]
\[ 2(12)(6) + 2(5) \frac{dy}{dt} = 0 \]
\[ \frac{dy}{dt} = \frac{-144}{10} \]
\[ \frac{dy}{dt} = -14.4 \text{ ft/sec} \]

Solution: -14.4 ft/sec

(6) Given the function \( f(x) = \frac{2x^2 - 2x - 12}{x^2 - 9} \), find the horizontal and vertical asymptotes.

Solution

First we simplify \( f(x) \):

\[ \frac{2x^2 - 2x - 12}{x^2 - 9} = \frac{2(x^2 - x - 6)}{(x - 3)(x + 3)} \]
\[ = \frac{2(x - 3)(x + 2)}{(x - 3)(x + 3)} \]
\[ = \frac{2(x + 2)}{x + 3} \]

HA:

\[ \lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{2(x + 2)}{x + 3} = 2 \]

VA:

\[ x + 3 = 0 \]
\[ x = -3 \]

(7) Given the function \( f(x) = 2x^4 + 16x^3 - 7 \),

(i) determine the critical numbers of \( f \).

Solution The domain of \( f \) is all real numbers. Then

\[ f'(x) = 8x^3 + 48x^2 \]

and so the domain of \( f' \) is all real numbers. Then

\[ 8x^3 + 48x^2 = 0 \]
\[ 8x^2(x + 6) = 0 \]
\[ 8x^2 = 0 \quad \text{or} \quad x + 6 = 0 \]

Therefore the critical numbers are \( x = 0, -6 \).

(ii) determine whether each critical number is a relative maximum, relative minimum, or neither.

Solution We have

\[ f'(\text{number less than -6}) = \text{negative value} \]
\[ f'(\text{number between -6 and 0}) = \text{positive value} \]
\[ f'(\text{number greater than 0}) = \text{positive value} \]

So \( f' \) is decreasing on \((-\infty, -6)\) and increasing on \((-6, 0)\) and \((0, \infty)\). Therefore, \((-6, f(-6)) \) gives a relative minimum and \((0, f(0)) \) will give neither a minimum nor a maximum.
(8) Use the function \( f(x) = 3x^5 - 10x^4 + x - 1 \) to

(i) find the inflection points of its graph.   (ii) determine where its graph is concave up.

(iii) determine where its graph is concave down.

Solution The first derivative is

\[ f'(x) = 15x^4 - 40x^3 + 1 \]

and the second is

\[ f''(x) = 60x^3 - 120x^2 = 60x^2(x - 2). \]

Solving \( 60x^2(x - 2) = 0 \), we find the “second-order critical points” as

\[ x_1 = 0, \quad x_2 = 2. \]

Verifying the signs of the second derivative on the intervals \( I_1 = (-\infty, 0), \ I_2 = (0, 2), \ \text{and} \ I_3 = (2, \infty) \), we obtain

- \( f''(-1) = -180 < 0 \). Therefore \( f(x) \) is concave down on \( I_1 \).  

- \( f''(1) = -60 < 0 \). Therefore \( f(x) \) is concave down on \( I_2 \). \( f(x) \) has the same concavity as on \( I_1 \) and thus \((0, -1)\) is not an inflection point.

- \( f''(3) = 540 > 0 \). Therefore \( f(x) \) concave up on \( I_3 \). Concavity is changing and thus \((2, f(2) = -79)\) is an inflection point.

(9) Find the following indefinite integrals:

(i) \[ \int \frac{x^3 + x^2 \sin(x) - 2}{x^2} \, dx \]

Solution

\[
\int \frac{x^3 + x^2 \sin(x) - 2}{x^2} \, dx = \int x \, dx + \int \sin(x) \, dx - 2 \int x^{-2} \, dx \\
= \frac{x^2}{2} - \cos(x) + \frac{2}{x} + C.
\]

(ii) \[ \int \frac{e^x}{e^x + 1} \, dx \]

Let \( u = e^x \). Then \( du = e^x \, dx \) and

\[
\int \frac{e^x}{e^x + 1} \, dx = \int \frac{du}{u+1} \\
= \ln |u + 1| + C \\
= \ln |e^x + 1| + C.
\]

Alternatively, let \( u = e^x + 1 \). Then \( du = e^x \, dx \) and

\[
\int \frac{e^x}{e^x + 1} \, dx = \int \frac{du}{u} \\
= \ln |u| + C \\
= \ln |e^x + 1| + C.
\]
(10) Evaluate the following definite integrals:

(i) \[ \int_{\frac{\pi}{8}}^{\frac{3\pi}{8}} \frac{3}{\cos^2(2x - \frac{\pi}{2})} \, dx \]

Solution

(i) Let \( u = 2x - \frac{\pi}{2} \). Then \( du = 2 \, dx \). Moreover,

\[
x = \frac{\pi}{4} \Rightarrow u = 0 \quad \text{and} \quad x = \frac{3\pi}{8} \Rightarrow u = \frac{\pi}{4}.
\]

Therefore,

\[
\int_{\pi/4}^{3\pi/8} \frac{3}{\cos^2(2x - \frac{\pi}{2})} \, dx = \frac{3}{2} \int_{0}^{\pi/4} \frac{du}{\cos^2(u)}
\]

\[= \frac{3}{2} \int_{0}^{\pi/4} \sec^2(u) \, du\]

\[= \frac{3}{2} \left[ \tan(u) \right]_0^{\pi/4}\]

\[= \frac{3}{2} (1 - 0)\]

\[= \frac{3}{2}.
\]

(ii) Let \( u = 1 - x^2 \). Then \( du = -2x \, dx \). Moreover,

\[
x = 0 \Rightarrow u = 1 \quad \text{and} \quad x = \frac{1}{2} \Rightarrow u = \frac{3}{4}.
\]

Therefore,

\[
\int_{0}^{\frac{1}{2}} \frac{x}{\sqrt{1 - x^2}} \, dx = \frac{-1}{2} \int_{1}^{\frac{3}{4}} \frac{du}{\sqrt{u}}
\]

\[= \frac{1}{2} \left[ \sqrt{u} \right]_1^{\frac{3}{4}}\]

\[= 1 - \sqrt{\frac{3}{4}}\]

\[= 1 - \frac{\sqrt{3}}{2}\]

\[= \frac{2 - \sqrt{3}}{2}.
\]
(11) For the subsequent questions, use the following definition and graph of \( f \).

\[
f(x) =
\begin{cases}
3 & \text{if } x < -3; \\
-1 & \text{if } x = -3; \\
2x + 2 & \text{if } -3 < x \leq 0; \\
x + 2 & \text{if } 0 < x < 3; \\
1 & \text{if } x = 3; \\
-2 & \text{if } 3 < x.
\end{cases}
\]

(i) What is \( \lim_{x \to -3^-} f(x) \)?

(ii) What is \( \int_{-2}^{2} f(x) dx \)?

(iii) What is \( \lim_{h \to 0} \left( \frac{\int_{0}^{2+h} f(x) dx - \int_{0}^{2} f(x) dx}{h} \right) \)?

Solution

(i) DNE because \( \lim_{x \to -3^+} f(x) = 2(-3) + 2 = -4 \neq 3 = \lim_{x \to -3^-} f(x) \)

(ii) \( \int_{-2}^{2} f(x) dx = \int_{-2}^{0} f(x) dx + \int_{0}^{2} f(x) dx = \int_{-2}^{0} (2x + 2) dx + \int_{0}^{2} (x + 2) dx = 6 \)

(iii) We have two approaches.

(I) \( \lim_{h \to 0} \frac{\int_{0}^{2+h} f(x) dx - \int_{0}^{2} f(x) dx}{h} = \lim_{h \to 0} \frac{\int_{2}^{2+h} f(x) dx}{h} = \lim_{h \to 0} \frac{1}{2} \frac{[f(2) + f(2 + h)] h}{h} = f(2) = 4 \)

where we have used the continuity of \( f \) at 2 and area of a trapezoid.

(II) \( \lim_{h \to 0} \frac{\int_{0}^{2+h} f(x) dx - \int_{0}^{2} f(x) dx}{h} = \lim_{h \to 0} \frac{g(2 + h) - g(2)}{h} = g'(2) = \left( \frac{d}{dx} \left[ \int_{0}^{x} f(t) dt \right] \right)_{x=2} = f(2) = 4 \)

where \( g(x) = \int_{0}^{x} f(t) dt. \)