Mathematics 2350, Calculus 3

FINAL EXAM, SPRING 2011

The use of calculator, formula sheet and/or any other electronic device is not allowed.

1. Let $\mathbf{F}(t)$ and $\mathbf{G}(t)$ be the vector functions

$$\mathbf{F}(t) = 3t^2\,\mathbf{i} - \sin t\,\mathbf{j} + e^t\,\mathbf{k}, \qquad \text{and} \qquad \mathbf{G}(t) = \cos(t)\,\mathbf{i} + \frac{1}{t^2}\,\mathbf{k}.$$

   Compute the dot product $\mathbf{F}(t) \cdot \mathbf{G}(t)$, and the cross product $\mathbf{F}(t) \times \mathbf{G}(t)$.

2. Let the velocity vector be $\mathbf{v}(t) = e^{2t}\mathbf{i} - \sin t\mathbf{j} + 2t\mathbf{k}$, and the initial position vector be $\mathbf{r}(0) = \mathbf{i} + \mathbf{2j} - \mathbf{k}$. Compute the acceleration vector $\mathbf{a}(t)$, and the position vector $\mathbf{r}(t)$.

3. Let the position vector be $\mathbf{R}(t) = e^t \cos t\,\mathbf{i} + e^t \sin t\,\mathbf{j}$. Find the unitary tangent vector $\mathbf{T}(t)$, the principal unit normal vector $\mathbf{N}(t)$, and the curvature $\kappa(t)$.

4. Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, if

$$xy^2 + yz^2 + zx^2 = 0.$$

5. Use the chain rule to find $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$, if

   $f(x, y) = \cos(xy)$, $x(u, v) = u + v$ and $y(u, v) = u - v$.

6. Suppose $f(x, y) = e^{2x+3y}$, $P = (1, 0)$ and $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.
   a) Find the directional derivative of f at P in the direction of $\mathbf{v}$.
   b) Find the maximum rate of change of f at P.

7. Find and classify all the critical points for the function

$$f(x, y) = 2x^3 - 6xy + 3y^2 + 5.$$

8. Solve the following integral by reversing the order of integration

$$I = \int_0^1 \int_{\sqrt{y}}^1 \sqrt{1 + x^3}\, dx dy.$$

9. Find the surface area of the part of the circular paraboloid $z = 8 - 2x^2 - 2y^2$ that lies inside the region $x \geq 0$, $y \geq 0$ and $z \geq 0$.

10. Find the volume of the tetrahedron bounded by the plane $4x + 2y + z = 4$ and by the coordinates planes $x = 0$, $y = 0$ and $z = 0$.

11. Evaluate the triple integral

$$I = \iiint_{\mathbf{D}} (3x^2 + 3y^2)\, dV,$$

where $\mathbf{D}$ is the region inside the cone $z = \sqrt{4x^2 + 4y^2}$, and below the plane $z = 8$.

12. Find div$\mathbf{F}$ and curl$\mathbf{F}$, where

$$\mathbf{F}(x, y, z) = e^x y\mathbf{i} + z \cos y\mathbf{j} + xy\mathbf{k}.$$

13. Let the vector field be $\mathbf{F} =< e^x y^2 + x^2, 2e^x y + 3y >$.
a) Verify that $\mathbf{F}$ is conservative.
b) Evaluate the scalar potential.
c) Find the line integral

$$\oint_C \mathbf{F} \cdot d\mathbf{R},$$

where C is the circle of radius 2 centered in the origin.

14. Use Green's theorem to evaluate

$$\oint_C (x \cos x - y)\, dx + \left(x + y^2 \tan^{-1} y\right) dy,$$

where C is the positively oriented rectangle with vertices in $(1, 0)$, $(3, 0)$, $(3, 1)$ and $(1, 1)$.

15. Evaluate the flux integral

$$\iint_S (\text{curl}\, \mathbf{F} \cdot \mathbf{N})\, dS,$$

where $\mathbf{F} = (z^2 + 3y)\mathbf{i} + (\cos z + 3x)\mathbf{j} + (xyz^2)\mathbf{k}$, S is part of the surface $\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$ with $z \geq 0$, and $\mathbf{N}$ points upward.