Mathematics 2450, Calculus 3 with applications

Fall 2015, version A

The use of calculator, formula sheet and/or any other electronic device is not allowed.

Multiple choice questions.

Follow the directions of the instructor.

1. Find the parametric equations for the line passing through the point \( P = (1, 2, 3) \) and perpendicular to the plane \( 3x - 2y + 5z = 4 \).

   a) \( t\mathbf{i} + 2t\mathbf{j} + 3t\mathbf{k} \)
   b) \( (3 + t, -2 + 2t, 5 + 3t) \)
   c) \( \frac{x - 3}{1} = \frac{y + 2}{2} = \frac{z - 5}{3} \)
   d) \( x + 2y + 3z = 14 \)
   e) \( (1 + 3t, 2 - 2t, 3 + 5t) \)

2. Let the velocity vector be \( \mathbf{v}(t) = t^2 \mathbf{i} + \cos t \mathbf{j} + e^{2t} \mathbf{k} \). Compute the acceleration vector \( \mathbf{a}(t) \).

   a) \( \left( \frac{t^3}{3} + c_1 \right) \mathbf{i} + (\sin t + c_2) \mathbf{j} + \left( \frac{1}{2} e^{2t} + c_3 \right) \mathbf{k} \)
   b) \( 2t \mathbf{i} - \sin t \mathbf{j} + 2e^{2t} \mathbf{k} \)
   c) \( \frac{t^3}{3} \mathbf{i} + \sin t \mathbf{j} + \frac{1}{2} e^{2t} \mathbf{k} \)
   d) \( 2t - \sin t + 2e^{2t} \)
   e) \( (2t + c_1) \mathbf{i} + (-\sin t + c_2) \mathbf{j} + (2e^{2t} + c_3) \mathbf{k} \)

3. Find the value of the following limit

   \[ A = \lim_{(x,y) \to (0,0)} \frac{x^2y}{x^4 + y^2}, \]

   if it exists.

   a) The limit does not exist
   b) \( A = \frac{1}{2} \)
   c) \( A = +\infty \)
   d) \( A = \frac{0}{0} \)
   e) \( A = 0 \)
4. Given $F(x, y) = \cos(xy)$ where $x = u^2 + v^2$ and $y = u^2 - v^2$. Use the chain rule (do not substitute for $x$ and $y$!) to find $\frac{\partial F}{\partial v}$. Express the result in terms of $x$, $y$, $u$, and $v$.

   a) $\frac{\partial F}{\partial v} = -\sin(xy)(y - x)2v$
   b) $\frac{\partial F}{\partial v} = -\sin(xy)(y + x)2u$
   c) The function is not differentiable
   d) $\frac{\partial F}{\partial v} = -\sin(xy)(2yu - 2xv)$
   e) $\frac{\partial F}{\partial v} = -\sin(xy)(2yv + 2xu)$

5. Let $f(x, y) = \cos(x + 3y)$, $P = \left(\frac{\pi}{2}, \frac{\pi}{3}\right)$ and $\mathbf{v} = -3\mathbf{i} + 4\mathbf{j}$. Find the directional derivative of $f$ at $P$ in the direction of $\mathbf{v}$.

   a) 9
   b) 0
   c) $\frac{9}{5}$
   d) $\frac{1}{5}(-3, 4)$
   e) $\langle 1, 3 \rangle$

6. Evaluate the integral by reversing the order of integration.

   $I = \int_0^1 \int_y^1 e^{x^2} \, dx \, dy$

   a) $I = 1$
   b) $I = e$
   c) $I = (e - 1)$
   d) $I = 0$
   e) $I = \frac{1}{2}(e - 1)$

7. Find the surface area of the portion of the cone $z = \sqrt{x^2 + y^2}$ inside the cylinder $x^2 + y^2 = 4$.

   a) $S = 4\sqrt{2}\pi$
   b) $S = 0$
   c) $S = 4\pi$
   d) $S = \frac{\pi}{6}(17^{3/2} - 1)$
   f) $S = \pi$
8. Find \( \text{curl} \, \mathbf{F} \), where
\[
\mathbf{F}(x, y, z) = (x^3 + 2x)\mathbf{i} + \cos(y)\mathbf{j} + e^{z^2}\mathbf{k}.
\]

a) \( \nabla \times \mathbf{F} = 3x^2 + 2 - \sin(y) + 2e^{z^2} \)

b) \( \nabla \times \mathbf{F} = \langle 3x^2 + 2, -\sin(y), 2e^{z^2} \rangle \)

c) \( \nabla \times \mathbf{F} = \langle 0, 0, 0 \rangle \)

d) \( \nabla \times \mathbf{F} = \sqrt{(3x^2 + 2)^2 + \sin^2(y) + (2e^{z^2})^2} \)

e) \( \nabla \times \mathbf{F} = 0 \)

9. Verify if the vector field \( \mathbf{F} = \langle x\cos(2y), -x^2\sin(2y) \rangle \) is conservative and evaluate the line integral
\[
I = \int_C \mathbf{F} \cdot d\mathbf{R},
\]
where \( C \) is the curve parametrized by \( \mathbf{R}(t) = \langle t, \pi t^2 \rangle \), for \( 0 \leq t \leq 1 \).

a) \( I = \frac{1}{4} \)

b) \( I = \frac{1}{2} \)

c) \( I = 0 \)

d) \( I = 1 \)

e) \( I = 2 \)

10. Use Green’s theorem to evaluate
\[
I = \oint_C (-y + y^2) \, dx + (x + 2xy) \, dy,
\]
where \( C \) is the rectangle with vertices in \((0, 0), (2, 0), (2, 1)\) and \((0, 1)\), traversed counterclockwise.

a) \( I = 4 \)

b) \( I = 0 \)

c) \( I = 1 \)

d) \( I = 2 \)

e) \( I = 8 \)

11. Use the divergence theorem to evaluate
\[
I = \iiint_S \mathbf{F} \cdot \mathbf{N} \, dS,
\]
where \( \mathbf{F} = \langle xz, yx, zy \rangle \), and \( \mathbf{N} \) is the unit outward normal to the surface \( S \) which encloses the box \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) and \( 0 \leq z \leq 1 \).

a) \( I = 3 \)

b) \( I = 0 \)

c) \( I = 1/2 \)

d) \( I = 3/2 \)

e) \( I = 1 \)
Essay questions.

Show all your work. A correct answer with no work counts as 0.

12. Let the position vector be \( \mathbf{R}(t) = 8t \mathbf{i} + 3\sin(2t) \mathbf{j} - 3\cos(2t) \mathbf{k} \). Find the unit tangent vector \( \mathbf{T}(t) \) and the principal unit normal vector \( \mathbf{N}(t) \).

13. Find and classify all the critical points for the function

\[
f(x, y) = 2x^3 + 3xy - 2y^3 + 7.
\]

14. Use either cylindrical or spherical coordinates to evaluate the triple integral

\[
I = \iiint_D z \, dV,
\]

where \( D \) is the portion of the ball, \( x^2 + y^2 + z^2 \leq 4 \), in the first octant, \( x \geq 0 \), \( y \geq 0 \) and \( z \geq 0 \).

15. Use Stokes’ theorem to evaluate the line integral \( \oint_C \mathbf{F} \cdot d\mathbf{R} \), where

\[
\mathbf{F} = (e^{x^2} + 3y) \mathbf{i} + (\cos y + x) \mathbf{j} + z^2 \mathbf{k}
\]

and \( C \) is the closed curve given by the line segments connecting the points \((1, 0, 0), (0, 1, 0), (0, 0, 1)\) traversed in the given order.