Limits to the fundamental measurement accuracy in Josephson junction array voltage intercomparisons

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The dc voltage output from a hysteretic Josephson junction which is locked to an ac frequency source differs from the ideal Josephson relation if the junction drives a current about a closed superconducting circuit. The difference in voltage ΔV from two hysteretic Josephson junctions driven in series opposition is proportional to the difference in their driving frequencies $\Delta\omega$ if the junctions are each biased to the nth voltage step. It is shown here, however, that ΔV is systematically smaller than the voltage difference ΔV_0 predicted by the ideal junction relation $\Delta V_0 = (n\hbar/2e)\Delta\omega$. If the loop inductance approaches zero, the smallest detectable voltage difference ΔV_0 between two junctions is limited by the intrinsic Josephson inductance. For arrays of more than one junction, however, ΔV_0 remains proportional to the loop inductance.

I. INTRODUCTION

Josephson junctions are extremely accurate frequency to voltage converters. Since 1972 the ac Josephson effect has been used as the method for voltage calibration and standardization.[1] In 1990, the Josephson junction array voltage standard became the basis for the internationally accepted standard volt. The array voltage standard consists of between 2076 and 18992 hysteretic Josephson junctions in series.[2–6] Each is bathed by the same microwave source. When the standard is operating at a constant dc voltage, each junction is locked to the microwave frequency at a zero-current crossing.

There has been experimental interest lately in investigating the accuracy with which the conversion from frequency to voltage occurs in these devices. In an experiment by Kautz and Lloyd,[7] two arrays were placed in series opposition and biased by the same frequency source, so that any discrepancies in frequency or voltage between the arrays would produce a dc current which may be detected in a low-inductance loop. They showed that two arrays, each with 3020 junctions, differed in voltage by no more than 2 parts in 10^{17}. Jain *et al.*[8] have used two junctions in series opposition but at different heights in order to confirm the absence of the gravitational red shift, and hence verified the equivalence principle between photons and Cooper-paired electrons. They found a measurement accuracy of 3 parts in 10^{19}. Such interjunction comparisons show the Josephson voltage standard to be a reliable device for detecting very small differences in frequency. In this letter we show that while small frequency differences may be detected in this manner to exceptional precision, they may not be accurately measured for hysteretic junctions operating near zero-current bias. We show that there is a systematic deviation from the ideal Josephson relation for hysteretic junctions as a consequence of the current in the superconducting measurement circuit. We shall quantify this deviation for parameters within the standard operating regime for Josephson junctions in the Josephson series-array voltage standard.

II. MODEL EXAMPLE

If a hysteretic Josephson junction with capacitance C is placed in a closed loop with inductance L, it acts as a pump, converting microwave energy into a dc current. In creating the current, the pump is clearly loaded, for work must be performed in building the accompanying magnetic field within the superconducting loop. The nature of this loading may be understood within the dynamics of the standard mechanical model in which the superconducting phase difference ϕ across the junction is analogous to the rotation angle of a physical pendulum. A junction which is phase locked to a constant voltage step is analogous to a driven pendulum which is locked to a fully rotating cycle. A closed superconducting circuit may be identified with a torsional spring which prevents the pendulum from rotating freely.[9] If the torsional spring is weak, the pendulum will remain effectively locked in a rotating state for many cycles. For a stable phase lock in the absence of noise, collapse from the rotating cycle will not occur until the potential energy stored in the torsional spring is roughly equivalent to the kinetic energy of rotation, or until the restoring torque of the torsional spring exceeds the maximum gravitational torque. The electrical analogy is as follows: A Josephson junction which is initially biased to the nth voltage step $V = (n\hbar/2e)\omega$ will remain locked until either (i) the potential energy stored in the magnetic field $(1/2L)(\hbar\phi/2e)^2$ is of the same order as the average energy stored in the junction capacitor $\frac{1}{2}C(\hbar\dot{\phi}/2e)^2$, or (ii) the current in the circuit exceeds the maximum current for the nth step. Here $\hbar$ is Planck's constant divided by 2π, and $2e$ is the charge of a Cooper pair. In this paper we have examined the system *before* the collapse of the rotating cycle. We find that before the rotational collapse, the junction appears to be locked to the driving frequency. However, its

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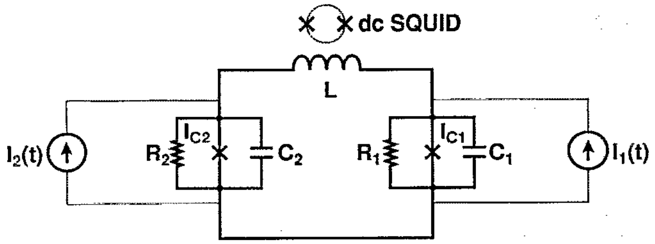

FIG. 1. The Stewart–McCumber model for two hysteretic Josephson junctions in series with an inductor L. The junctions, numbered 1 and 2, are subjected to microwave drives $I_1(t)$ and $I_2(t)$, respectively. Each junction has an associated shunt resistance R and capacitance C.

phase evolves at a slightly lower frequency, and thus it delivers an effectively lower voltage. To illustrate this point, we consider the case of two identical Josephson junctions biased to the same voltage step by slightly different frequency sources. The junctions are placed in series opposition in a closed superconducting loop with inductance L. Within the Stewart–McCumber model,[10] the junctions may be modeled by the circuit shown in Fig. 1. The phase differences across junction #1 and junction #2, ϕ_1 and ϕ_2, respectively, obey the following coupled equations of motion for zero current biases:

$$\ddot{\phi}_1+\frac{1}{RC}\dot{\phi}_1+\frac{1}{LC}(\phi_1-\phi_2)+\frac{2e}{\hbar C}I_0\sin(\phi_1)$$
$$=\frac{2e}{\hbar C}I_1\cos(\omega_1 t+\delta_1), \tag{1}$$

$$\ddot{\phi}_2+\frac{1}{RC}\dot{\phi}_2+\frac{1}{LC}(\phi_2-\phi_1)+\frac{2e}{\hbar C}I_0\sin(\phi_2)$$
$$=\frac{2e}{\hbar C}I_2\cos(\omega_2 t+\delta_2). \tag{2}$$

The junction shunt resistance is R, the junction critical current is I_0, and the driving amplitude for junctions #1 and #2 is I_1 and I_2, respectively. For simplicity in what follows we will take I_1 and I_2 to be equal, and we take the phases δ_1 and δ_2 to be zero. Typical junction parameters are as follows: $R=10\ \Omega$, $C=20$ pF, $I_0=200\ \mu$A, $\omega/2\pi=96$ GHz, and $I_1=20$ mA.[5] A typical measurement-loop inductance L is 2 μH.[7] If we rescale time in multiples of the driving period, Eqs. (1) and (2) take on the dimensionless form:

$$\ddot{\phi}_1+\gamma\dot{\phi}_1+\omega_0^2(\phi_1-\phi_2)+A\sin(\phi_1)+B\cos(\Omega_1 t), \tag{3}$$

$$\ddot{\phi}_2+\gamma\dot{\phi}_2+\omega_0^2(\phi_2-\phi_1)+A\sin(\phi_2)=B\cos(\Omega_2 t), \tag{4}$$

where

$$\gamma=\frac{1}{RC\omega}=10^{-2},\quad \omega_0^2=\frac{1}{LC\omega^2}=10^{-7},$$

$$A=\frac{2e}{\hbar C\omega^2}I_0=10^{-1},\quad B=\frac{2e}{\hbar C\omega^2}I_{1(2)}=10,$$

and

$$\Omega_{1(2)}=\frac{\omega_{1(2)}}{\omega}\cong 1.$$

We notice in (3) and (4) that the two junctions are coupled together by a harmonic force which depends on the sum of the phases. If both junctions are driven at the same frequency in opposite "directions", the coupling term $\omega_0^2(\phi_1-\phi_2)$ is zero on average, and is insignificant for typical values of ω_0^2. However, as we will see below, the effect of the coupling may not be neglected when the driving frequencies are different. Because we are interested in studying the effect of the coupling for small frequency differences, in what follows we will treat the coupling through ω_0^2 perturbatively.

We begin by treating the zeroth-order (uncoupled) system. When ω_0^2 is zero, the junctions are independent, and the evolution of the phase ϕ for either junction #1 or #2 is determined by the interplay of the frequencies of the periodic driving term and the periodic potential $A\cos(\phi)$. Locking occurs such that on the average, $\dot{\phi}_{1(2)}=+(-)n\Omega_{1(2)}$, where n is an integer (or interger fraction, but we shall be unconcerned with these cases). When the frequencies Ω_1 and Ω_2 are slightly different, the difference of the phases grows linearly with time such that on the average $\phi_1-\phi_2=n(\Omega_1-\Omega_2)t$. In the zeroth-order approximation, one would expect to observe a linearly increasing dc current of magnitude

$$I=\frac{1}{L}\Delta V_0 t, \tag{5}$$

where

$$\Delta V_0=n\frac{\hbar}{2e}(\Omega_1-\Omega_2). \tag{6}$$

However, we note that for nonzero ω_0^2 the potential energy is no longer purely sinusoidal; the periodic force becomes "tilted" by the harmonic restoring force. To study the effect of this tilting to first order in ω_0^2, we assume solutions to (3) and (4) of the form

$$\phi_1(t)=\phi_1^0(t)+x_1(t), \tag{7}$$

$$\phi_2(t)=\phi_2^0(t)+x_2(t), \tag{8}$$

where $\phi_{1(2)}^0(t)$ is the steady-state zeroth-order locked solution, and $x_{1(2)}(t)$ is the deviation caused by the coupling ω_0^2. The junctions are operating in the regime in which $A/\Omega^2\ll 1$. In such a case, the evolution of the phase in zeroth-order is approximately

$$\phi_1^0(t)=n\Omega_1 t-\frac{B}{\Omega_1^2}\cos(\Omega_1 t)+\phi_1(0), \tag{9}$$

$$\phi_2^0(t)=n\Omega_2 t-\frac{B}{\Omega_2^2}\cos(\Omega_2 t)+\phi_2(0), \tag{10}$$

where $\phi_1(0)$ and $\phi_2(0)$ are initial phases. In writing (9) and (10), we have also neglected terms of order γ/Ω. The zeroth-order solutions require that at steady-state the damping force be counteracted by the driving force resonance such that

$$\gamma n\Omega_1 = AJ_n(B/\Omega_1^2)\sin\left(\frac{n\pi}{2}+\phi_1(0)\right), \tag{11}$$

$$\gamma n\Omega_2 = AJ_n(B/\Omega_2^2)\sin\left(\frac{n\pi}{2}+\phi_2(0)\right), \tag{12}$$

where $J_n(x)$ is the ordinary Bessel function of the first kind of order n.[11] On substituting (7) and (8) with (9) and (10) into (3) and (4), and expanding to linear order in x_1 and x_2, we obtain coupled equations for x_1 and x_2:

$$\ddot{x}_1+\gamma\dot{x}_1+\omega_0^2 x_1 + A\cos[\phi_1^0(t)]x_1 = \omega_0^2 x_2 - \omega_0^2[\phi_1^0(t)-\phi_2^0(t)] - Af_1(t), \tag{13}$$

$$\ddot{x}_2+\gamma\dot{x}_2+\omega_0^2 x_2 + A\cos[\phi_2^0(t)]x_2 = \omega_0^2 x_1 - \omega_0^2[\phi_2^0(t)-\phi_1^0(t)] - Af_2(t). \tag{14}$$

In Eqs. (13) and (14), $f_{1(2)}(t)$ is the oscillating part of $\sin[\phi_{1(2)}(t)]$. In solving (13) and (14) to lowest order in A/Ω^2, it is sufficient to replace the parametric term $A\cos[\phi^0(t)]$ by its time average:

$$\langle A\cos[\phi^0(t)]\rangle = AJ_{-n}(B/\Omega^2)\cos\left(\frac{n\pi}{2}+\phi(0)\right) = \pm\sqrt{A^2J_n^2(B/\Omega^2)-\gamma^2n^2\Omega^2}. \tag{15}$$

In this approximation, Eqs. (13) and (14) describe two coupled oscillators which are displaced on the average by a force with linearly increasing magnitude $\omega_0^2 \times n(\Omega_2-\Omega_1)t$. Consequently the time averages of $\dot{x}_1$ and $\dot{x}_2$ are nonvanishing. This is important, for it indicates that the dc voltage difference between the two junctions will show deviations from the ideal voltage difference ΔV_0. We obtain the deviation in the dc voltage difference, $\delta(\Delta V_0)$, after solving (13) and (14) for the average of the difference of the velocities $\dot{x}_1-\dot{x}_2$. We find that

$$\delta(\Delta V_0) = -\Delta V_0 \times \omega_0^2 \frac{(P_2^2+P_1^2-2\omega_0^2)}{(P_2^2P_1^2-\omega_0^4)}, \tag{16}$$

where

$$P_{1(2)}^2 = \omega_0^2 \pm \sqrt{A^2J_n^2(B/\Omega_{1(2)}^2)-\gamma^2n^2\Omega_{1(2)}^2}. \tag{17}$$

We choose the positive sign to assure a stable phase lock for small ω_0^2. When the difference in frequencies approaches zero, (16) may be simplified, with the result that

$$\delta(\Delta V_0) = -\Delta V_0 \frac{2\omega_0^2}{2\omega_0^2+\sqrt{A^2J_n^2(B)-\gamma^2n^2}}. \tag{18}$$

Equation (18) is the central result of this paper. It is valid provided that time is short enough so that $|x_{1(2)}|<1$, which implies that t is less than the time for the current to exceed its maximum for the nth voltage step:

$$t < \frac{\hbar}{e|\Delta V_0|} \times \frac{2\omega_0^2+\sqrt{A^2J_n^2(B)-\gamma^2n^2}}{2\omega_0^2}. \tag{19}$$

We see from (18) that the difference in voltage is suppressed from its ideal value ΔV_0. For smaller loop inductance the suppression increases; the measured voltage difference approaches zero with the loop inductance. We have performed numerical simulations of Eqs. (3) and (4) in order to demonstrate the applicability of the perturbative solution. As shown in Fig. 2, the numerical results coincide with the predictions of Eq. (18).

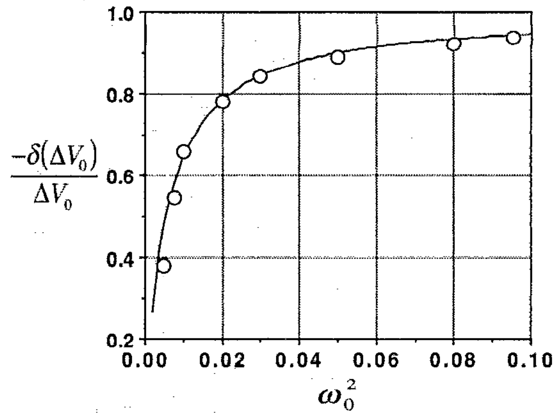

FIG. 2. The fractional voltage deviation $\delta(\Delta V_0)/\Delta V_0$ as a function of the LC frequency ω_0^2 for two opposing junctions connected by a superconducting loop with inductance L. The data were obtained from a computer simulation of the circuit in Fig. 1, and the smooth curve is the prediction of expression (18). The junction parameters were selected as follows: $n=1$, $\Omega_1=1$, $\Omega_2=0.9995$, $A=0.1$, $B=0.3$, and $\gamma=0.01$. We observe that the deviation from the ideal voltage difference increases as the loop inductance L decreases.

The same analysis may also be applied to a circuit containing two opposing multiple-junction array voltage standards. The calculation of the deviation from the ideal voltage difference is straightforward, even if the individual junctions within the array are nonidentical and biased to different voltage steps. It is most illustrative, however, to consider a simplification in which (i) the two arrays are identical, (ii) there are N identical junctions in each array, and (iii) each junction is biased to the nth voltage step (in the same direction). In this case, the analogue of (18) is

$$\delta(\Delta V_0) = -\Delta V_0 \frac{2\omega_0^2}{2\omega_0^2 N+\sqrt{A^2J_n^2(B)-\gamma^2n^2}} \tag{20}$$

provided that

$$t < N\frac{\hbar}{e|\Delta V_0|} \times \frac{2N\omega_0^2+\sqrt{A^2J_n^2(B)-\gamma^2n^2}}{2\omega_0^2}. \tag{21}$$

In (20), the voltage difference $\Delta V_0 = (\hbar/2e)Nn(\omega_1-\omega_2)$. We see that as the loop inductance approaches zero the percentage of deviation from the ideal voltage difference is no longer as large as it was in the case of only two junctions. While the maximum deviation in the voltage is still equal to the voltage difference between two junctions, this is only $1/N$ times the total voltage difference between the two arrays. The N junctions share the burden of building the magnetic field associated with the current loop. Each junction is affected less, and there is therefore a proportionally smaller voltage deviation.

III. DISCUSSION: EFFECT ON THE MEASURED dc CURRENT

We have seen that the actual voltage difference will be suppressed from the ideal voltage difference in a junction intercomparison. This implies also that the average current in the loop will be less than ideal. In fact, for two junctions driven by slightly different frequencies, the buildup of dc current in the measurement loop is given by

$$I=\frac{1}{L}\Delta Vt=\frac{\Delta V_0}{L}\left(\frac{\sqrt{A^2J_n^2(B)-\gamma^2n^2}}{2\omega_0^2+\sqrt{A^2J_n^2(B)-\gamma^2n^2}}\right)t. \qquad (22)$$

We note that as a function of the inductance, expression (22) has two limits. When the inductance is large, so that $\omega_0^2 \ll A$, we recover expression (5); the dc current induced in the measuring loop is inversely proportional to the loop inductance L. On the other hand, when the inductance is small enough that $\omega_0^2 \gg A$, the voltage difference ΔV is proportional to the loop inductance L, and in the limit in which L approaches zero, the current is independent of L

$$I=\frac{e}{\hbar}I_0\Delta V_0 t\times\sqrt{J_n^2(B)-\left(\frac{\hbar\omega}{2e}\right)^2\frac{n^2}{I_0^2R^2}}. \qquad (23)$$

We observe in (23) that the dc current is limited by an effective inductance L_i:

$$L_i=2L_J\times\left[\sqrt{J_n^2(B)-\left(\frac{\hbar\omega}{2e}\right)^2\frac{n^2}{I_0^2R^2}}\right]^{-1}, \qquad (24)$$

where L_J is the Josephson inductance $(\hbar/2eI_0)\sim 10$ pH. A typical dc superconducting quantum interference device (SQUID) creates a current noise figure $\Delta I \sim I_n/\sqrt{\text{Hz}}$, which is 10^{-13} A/$\sqrt{\text{Hz}}$. If the bandwidth is approximately the reciprocal of the observation time t, then the signal will rise above the noise when t is long enough such that

$$t>\left(\frac{L_iI_n}{\Delta V_0\sqrt{\text{Hz}}}\right)^{2/3}. \qquad (25)$$

However, t cannot be longer than the time for phase unlocking, which is set by (19) in the limit of zero inductance. Consequently, for a given observation time, the detectable voltage difference is bounded above and below by the two limits expressed in (19) and (25), respectively. For example, for an observation time of 10 min, a voltage difference ΔV_0 between 10^{-18} and 10^{-28} V can be detected (the lower bound of 10^{-28} V is pedantic because quantum fluctuations limit the sensitivity below about 10^{-22} V). Such limitations on the accuracy for finite observation times are well known. However, the analysis here shows that when the Josephson inductance is much greater than the loop inductance, the limits are practically independent of loop inductance.

Let us consider now the dc current in the case of two arrays with N identical junctions, all identically biased, such that the deviation from the ideal voltage is given by (20). The loop current will go as

$$I=\frac{\Delta V_0}{L}\left(\frac{2(N-1)\omega_0^2+\sqrt{A^2J_n^2(B)-\gamma^2n^2}}{2N\omega_0^2+\sqrt{A^2J_n^2(B)-\gamma^2n^2}}\right)t. \qquad (26)$$

For L less than L_i/N the voltage deviation is greatest, and we observe that

$$I\sim\frac{\Delta V_0 t}{L[N/(N-1)]}. \qquad (27)$$

In this limit the current depends on the loop inductance but not on the Josephson inductance. Within this model, therefore, we have shown that it is advantageous to use an array of Josephson junctions for the detection of small frequency and voltage differences; the inductance may be made arbitrarily small to improve the detection of small voltage differences. In contrast, for only two junctions there is practically no improvement in the signal for a loop inductance much less than L_J.

As we have already mentioned, two junctions connected in series will remain locked until the potential energy stored in the loop inductance $\frac{1}{2L}[\hbar(\phi_1-\phi_2)/2e]^2$ is of the same order as the average energy stored in each junction capacitor $\frac{1}{2}C(\hbar\dot{\phi}_{1(2)}/2e)^2$ or until the current in the circuit exceeds the maximum current for the nth step, whichever occurs first. For $L=1$ nH and $\Delta V_0=10^{-20}$ V, the time for energies to match is about 10^6 s, while the time to reach the critical current is ten times larger. The same is true for two opposing arrays of N junctions. These times are several orders of magnitude longer than typical measurement times, so it is not likely that systematic phase unlocking will occur for these parameters. On the other hand, we have neglected the effect of random fluctuations in this model, as well as the systematic oscillations of the drive terms in (13) and (14). As the loop inductance is made smaller, the effects of these ac sources may be enhanced and lead to phase unlocking. We address here the affect of the driving oscillations; the characterization of the coupled system in the presence of noise will be studied in a forthcoming article. For (9) and (10) we have assumed phases such that the oscillating terms will cancel at $t=0$. They become important after t on the order of $\hbar/2e\Delta V_0\sim 10^5$ s for ΔV_0 of 10^{-20} V. In actual junction operation however, the oscillations need not cancel, in which case they may completely inhibit phase locking when the loop inductance is small. In order that this not be the case, the response to the drive in (13) and (14) must be less than 1;

$$\omega_0^2B/|1-2\omega_0^2-\sqrt{A^2J_n^2(B)-\gamma^2n^2}|\lesssim 1. \qquad (28)$$

For $B=10$ and $A=0.1$ this implies that $L>60$ pH, which is still greater than L_i for these junction parameters. Only for $B<2$ can the loop inductance be made arbitrarily small without immediately causing instabilities.

In summary, we have quantified the effects of a closed current loop on the operation of hysteretic Josephson junctions at zero current bias. We have analyzed a particular situation in which two junctions are placed in series and driven in opposition by slightly different frequencies. We have shown that the difference in their voltages is smaller than that predicted by the ideal Josephson relation for isolated junctions. The deviation from ideal behavior is important in estimating the minimum voltage difference which can be detected in a finite observation time. In the absence of noise, we find that the smallest detectable voltage difference between two single hysteretic junctions is limited by the intrinsic Josephson inductance. For multiple junction arrays, however, the smallest detectable voltage is limited by the inductance in the measurement loop.

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