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To prove that the general term in the development in a series of powers of a of the reciprocal of $(1-a)(1-ax)\ldots(1-ax^i)$ (say of Fx) is $(1-x^{j+1})(1-x^{j+2})\ldots(1-x^{j+i})\div(1-x)(1-x^2)\ldots(1-x^i).a^j$ say $J_x a^j$, I proceed as follows.

I call the coefficient of a^j in the development X_j, and show that every linear factor of X_j is contained in J_x.

Any such factor, say $(x-\rho_j)$, is a primitive factor of x^r-1, where r is any integer such that $E\frac{i+j}{r}-E\frac{i}{r}-E\frac{j}{r}=1$ and it is unrepeated.

Let $x=\rho$, and let the negative minimum residue of $i-1$ in respect to r be $-\delta$.

Then $F\rho$ is equal to the product of δ linear functions of a divided by a power of $(1-a^r)$, and consequently the only powers of a (say a^θ) which appear in its development will be those for which the residue of θ in respect to r, is 0, 1, 2, ... δ, and consequently $E\frac{i+\theta}{r}-E\frac{i}{r}-E\frac{\theta}{r}=0$.

Hence a^j will not appear therein: so that X_j vanishes when any factor of J_x is zero, and consequently since every such factor is unrepeated X_j contains J_x.

But J_x is obviously of the degree ij in x, and X_j which is the sum of the j-ary homogenous products of $1, x, x^2, \ldots, x^i$ is of the same degree. Hence the two functions of x can only differ by a constant factor. On making $x=1$, Fx becomes $(x-a)^{-i}$; so that X_j becomes

$$\frac{(j+1)(j+2)\ldots(j+i)}{2\ldots i}$$

and J_x becomes the product of vanishing fractions

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Hence $X_j=J_x$. Q. E. D.

The expansion of $(1-ax)(1-ax^2)\ldots(1-ax^i)$ in terms of powers of a may be verified in like manner.

It is not without interest to observe (if the remark has not been made before) how this development is connected by the principle of correspondence with the preceding one.

Throwing out by multiplication the factor $(1-a)$ in the denominator of Fx we obtain the reciprocal of $(1-ax)(1-ax^2)\ldots(1-ax^i)$, say $\frac{1}{Gx}$ under the form

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When $i=\infty$ each of these developments (like a multitude of others, including the Theta-functions) may be obtained intuitively by the graphical method of points given in my communication to the Johns Hopkins Scientific Association at its last meeting; it remains a desideratum to apply the same method to the above two developments, or either of them, for the case of i.*

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