

CRITICISMS AND DISCUSSIONS.

THE THEORY OF MATHEMATICAL FORM.

A CORRECTION AND EXPLANATION.

In his interesting communication to *The Monist* of January 1894 on "The Logic of Relatives," Mr. C. S. Peirce, while alluding in flattering terms to my "Memoir on the Theory of Mathematical Form," takes exception to the opinion, which he conceives to be there put forward, that "a relationship" is "nothing but a complex of bare connexions of pairs of objects," and accordingly states that "while I have learned much from the study of Mr. Kempe's Memoir, I am obliged to modify what I have found there so much that it will not be convenient to cite it, because long explanations of the relation of my views to his would become necessary if I did so."

Any criticism which comes from so distinguished a logician as Mr. Peirce must of course have great weight; but in the present instance I fear that *bonus dormitat Homerus*, for I am confident that I have never held or expressed, either directly or by implication, any such opinion as he attributes to me. On the contrary, I regard it as quite inconsistent with the fundamental principles formulated in the Memoir. I am fully alive to the many defects of my essay, and am glad of this opportunity of expressing my gratitude to Mr. Peirce for a long and valuable letter of criticism, which he sent me on January 17, 1887, a letter which led to my inserting a "Note" in the *Proceedings of the Royal Society*, Vol. 42 (1887), p. 193, containing some very necessary corrections and emendations of the Memoir. But as that Memoir has now for the second time been called to the attention of the readers of *The Monist*, Mr. F. C. Russell having referred to it with appreciation in the issue of April, 1894, I have become solicitous to maintain its reputation here at as high a pitch as possible, and am anxious, therefore, that no undeserved criticism should pass unnoticed. Mr. Peirce's article fortunately affords me abundant evidence wherein it is that he has mistaken my views, and with the permission of the Editor I propose to indicate the nature of his misconception. In doing this I may be pardoned if I seize the opportunity to state concisely and without reference to details,

exactly what the fundamental principles set forth in the Memoir are. Unless I did so, what I have to say would be scarcely intelligible.

It must be obvious to the most casual observer that the subject-matter of thought consists of much which may be dismissed from consideration without affecting those properties which it is the object of the logician or mathematician to investigate. The determination in any case of how much may be thus dismissed, of what the irreducible minimum which must be retained consists, and of the causes to which the properties of this essential residue are due, is, however, by no means simple. To effect such determination, is the object of the Memoir. It points out, in the first place, that the subject-matter of exact thought is conceived of and dealt with by the mind in the processes of reasoning as consisting of a number of separate things, as being a plurality of "units." These "units" are of every conceivable description. They may be "individuals or abstract ideas,"—to quote the words of Mr. Peirce,—or, to quote my own (Memoir, par. 4), may be "material objects, intervals or periods of time, processes of thought, points, lines, statements, relationships, arrangements, algebraical expressions, operations, operators," etc., etc. The task of specifying the exact objects or ideas which are thus conceived of and dealt with by the mind in any investigation, may in some cases be one of extreme difficulty, and mistakes are likely to occur unless the operation be conducted with great care. Thus, while no difficulty may be experienced in enumerating this or that object or idea as units which are under consideration, there may be a failure to appreciate that their "relations" are also in certain cases being conceived of and dealt with as units, and should be included in the enumeration; that the system of units before us is consequently a more extensive one than is supposed, and that unless the whole field of view is brought into focus, an erroneous impression may be created.

But, while it is important that we should not overlook any of the "units" which are really before us, it is equally important to notice that, for the purpose of defining or investigating a particular subject-matter, it is frequently useful to introduce into our field of view certain conceptions which, though valuable auxiliaries in such definition and investigation, and on occasion themselves the subject of our study, yet form no essential part of the particular subject-matter for the time being under consideration. The "relationships" existing between the objects which compose that subject-matter, the "operations" by which one individual or plurality of those objects may be conceived of as derived from another individual or plurality, and "statements" with reference to such relationships or operations are instances of "auxiliary units" which may be thus introduced. Here it may be necessary to bear in mind that such units are merely auxiliary, that they are used merely as a temporary scaffolding, which, however useful during the course of our work, may obscure the true proportions of the structure which it environs.

But probably the greatest caution is needed with regard to a somewhat subtle danger, viz., the regarding of a number of separate things or conceptions, even in-

finite in number, as if they were not many, not a plurality, but one thing or one conception only. We have, for example, a number of objects before us, and we have certain relations between pluralities of those objects also before us, and we appreciate the fact that we are dealing with these relations as units. Where the relation between the individuals of one plurality is different from that between the individuals of another, we also appreciate the fact that the former relation regarded as a unit must differ from the latter relation when so considered, and that consequently we have *two* relations before us, and not *one*. But where the relation between the units of one plurality is, in common parlance, "the same as" that between the units of another, where the two relations are said to be "identical," there are certainly many persons who would come to the conclusion that here we have but *one* relation before us, not *two*. The conclusion would, however, be a wrong one, the use of the words "the same" and "identical," though by no means uncommon, and in general not likely to lead to confusion, being here erroneous. The two relations are no doubt "undistinguishably alike," so that "one single description will apply equally to either of them" (Whately's Logic, first edition, p. 298 Appendix on Ambiguous Terms), but this is not "identity," this is not being "the same," there are *two* relations before us, and not *one*, and the failure to appreciate the fact would be to overlook a matter of fundamental importance.¹

To follow this up, let me add that, in the remarks which ensue, when I say that two or more objects or conceptions differ from each other, I do not mean simply that they are a plurality and not one object or conception only, but that they are unlike in some respect, that something can be said of one that is not true of the other or others. On the other hand, when I say that certain objects or conceptions (in the plural) differ in no respect whatever, I do not mean that the use of the plural is erroneous and we have really only one object or conception in view, but I mean that they compose a plurality of objects or conceptions each of which is undistinguishably like each of the others, so that a description of one is equally applicable to each of those others.

The subject-matter of exact thought consists then of a mass of "units." These are not, however, conceived of by the mind as jumbled together in a mere confused heap, but exhibit a certain orderly arrangement which I term "mathematical form." What exactly is this "form," and to what is it due?

In answering this question let me ask my readers to take a somewhat general view of the subject-matter of thought. From whatever point of view we regard this, the most prominent feature is probably the combination of variety and uni-

¹ Great confusion of ideas is often produced, and many fallacies engendered in otherwise enlightened minds by not being sufficiently alive to the fact (in itself not always to be avoided), that they use the same name to express ideas so different as those of identity and undistinguishable resemblance." J. S. Mill, *Logic*, second edition, Vol. I, p. 93.

² The two following paragraphs are taken almost *verbatim* from the writer's Presidential Address to the London Mathematical Society on "Mathematics" of November 8th, 1894. See *Proc. Lond. Math. Soc.*, Vol. XXVI, p. 5.

formity which it exhibits. We picture to ourselves things of every imaginable description, differing in every possible way; but the representation includes objects here few in number, there many, which do not differ from each other in any respect whatever. A still greater variety is displayed by the pluralities of the things pictured, the differences between which depend not only upon the number and peculiarities of the individual objects of which each consists, but also on those additional characteristics of unlimited diversity, which accompany every plurality, and may be concisely referred to as "relations."¹ Here again, though difference is present, its absence is equally marked; and pluralities, however complex, are rarely unique but are accompanied by other pluralities, some by one, some by more, to which they bear an undistinguishable resemblance.

Now putting aside the contemplation of the special peculiarities and characteristics of the individuals and relations which come under our observation in such infinite variety, let us confine our attention to the study of the results which flow from the mere fact that this and that individual, or this and that plurality, differ, while this and that do not. We shall not, as might at first sight be supposed, thereby put away everything which gives life and interest to the subject-matter of our thought and leave nothing but a mere heap of dry bones. *Form* remains. The like and unlike individuals and pluralities which are contained in any greater plurality must be distributed in some way through the whole body of individuals composing that greater plurality, and the way in which this distribution is effected gives to the latter a characteristic "form," which may or may not be different in two pluralities of the same number of individuals.

It is this "form" which I believe to constitute all that is essential in the subject-matter of thought so far as the processes of reasoning are concerned. When the "form" is determined, those "properties" and "relations" of the subject-matter which are the study of the mathematician and logician are also determined. The rest is mere dress. It follows, of course, that two subject-matters which are of like "form," however widely they may differ in other respects, will for the exact thinker have precisely similar properties and admit of precisely similar treatment.

So much for the general theory. Now let us turn to the special matter upon which Mr. Peirce has misapprehended me. For the definition of the "form" of any system of units it is necessary only to indicate which of those units, and which of the pluralities of those units, differ from each other and which do not. In order to do this it will not in general be necessary to make the indication for each separate unit and plurality. The distribution of the various differing and non-differing

¹ Differences may exist between two pluralities though none exists between the individuals which compose them. Thus geometrical points differ in no respect from each other, but a collinear triad of such points compose a plurality which differs from the plurality composed of a triad of non-collinear points.

² See "The Subject-Matter of Exact Thought," by the writer in *Nature*, Vol. 43 (1890), p. 156, where the geometrical theory of points and the logical theory of statements are considered and compared in the light of the principles here set forth.

units and pluralities of any system is regulated by definite laws, so that a knowledge of the mode of distribution of some only of these components may determine that of the rest, and consequently the form of the system. There are in general several ways in which the form of a given system may be thus determined, and accordingly various different definitions of the same system may be adopted. There are certain systems the "forms" of which are fully defined when we know (1) which of their component units differ and which do not, and (2) that certain specified pairs of those units differ from the rest. The pairs may be of many different sorts, and so may the larger pluralities, but the existence of all the differences involved is fully indicated by the presence and absence of the differences between the units, and by the dichotomy of the pairs. As has already been stated, the definition of the "form" of a subject-matter may be facilitated by the introduction into our field of view of "auxiliary units." Now it is demonstrated in the Memoir that, by the introduction of a proper system of auxiliary units, an enlarged system is obtained of the special description just alluded to. The "form" of this enlarged system being determined, that of the smaller one obtained by omitting the auxiliary units is also known, and consequently is defined by the limited specifications I have mentioned. This result is a very important one in reference to the graphical representation of the "form" of any subject-matter of exact thought, for it enables us to effect such representation by means of a diagram consisting only of spots and of lines connecting certain of these spots, one to one. The spots represent the units, unlike spots representing units which differ, like spots those which do not. The various units of the system under consideration, and also the necessary auxiliary units, being duly represented, each by its representative spot, we have then merely to effect a proper division of the pairs of these spots into two sets. This may be done by connecting the two spots of each pair of the one set by a line, no lines being drawn in the case of the pairs of the other set. It matters not which of the two sets of pairs consists of the joined pairs, and which of the unjoined pairs for the lines are not employed to represent anything in the nature of a connexion between the units represented by the spots which are connected by those lines. All that is needed is that all the pairs of one set should be joined pairs, and all of the other unjoined, and that the pairs should thus be made to differ from each other.

Any subject-matter of exact thought admits then, I say, of representation by a diagram consisting of spots, and of lines connecting them in the manner indicated. Such diagrams are, however, sometimes used in an entirely different way from that in which they are used in the Memoir. The chemical diagrams showing the constitution of compounds, the "graphs" of Clifford and Sylvester, and those of Mr. Peirce in his article are instances. In these graphs the lines or bonds represent a "connexion," a "copula," a "mode of junction" between the things represented by the spots. It is quite comprehensible therefore that Mr. Peirce having such use of these bonds in mind, should have supposed that "in Mr. Kempe's method the spots represent the objects, whether individuals or abstract ideas, while the bonds

represent the relations," and that consequently "Mr. Kempe seems to consider relationship to be nothing but a complex of bare connexions of pairs of objects." It will, however, I hope, be clear from the foregoing explanation, and a reference to the Memoir itself, that I do not use, and have not there used, these lines or bonds as representing any relationship in the nature of a "connexion," but simply to distinguish certain pairs of things from others. And it will, I trust, be equally clear that because I believe that the "form" of any subject-matter may be defined by statements as to the existence and non-existence of differences between "units," and as to the existence of differences between pairs of those units, it does not follow that I believe that the complex relationships between pluralities of units which result from the possession of that form, are nothing but "complexes of bare connexions of pairs of objects."

A. V. KEMPE.

LONDON.

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