VOL. XVI. OCTOBER, 1906. No. 4

THE MONIST

PRAGMATISM AND MATHEMATICAL LOGIC.*

IT is certainly not one of the least of the merits of the *Leonardo* that it has established lines of communication and encouraged the exchange of ideas between exponents of philosophical studies belonging to the most diverse and distant intellectual fields—between logicians and estheticians, between moralists and economists, mathematicians and mystics, biologists and poets.

Pending the possibility of a comparative examination of the results obtained, or prepared, by the development and exchange of ideas in all these various directions, it will not be irrelevant to summarize here in a schematic synopsis such of these results as relate to one of the most important lines of communication which the *Leonardo* has helped to construct and keep in operation, that is, the line which joins the various domains of Pragmatism with those occupied and cultivated by the "mathematical logicians." A significant indication of the intimate connection between these two fields of philosophical research may be deduced from the fact that the sponsor of the denomination and concept of "Pragmatism" (Ch. S. Peirce) is himself likewise the initiator and promoter of an original trend in logico-mathematical studies.

It is not, however, from the labors of the school of

* Translated from the Italian by H. D. Austin. This paper appeared originally in the *Leonardo*, Feb., 1906.

Peirce, but rather from those of the Italian school, headed by Peano, that I purpose here to take my material for the determination of what might be called the "pragmatic characteristics" of the new logical theories.

* * *

One point of contact between logic and pragmatism is found in their common tendency to regard the value, and even the meaning, of every assertion as being intimately related to the use which can be made, or which it may be desired to make, of it for the deduction and construction of particular consequences or groups of consequences. This tendency is manifest, among the mathematical logicians, especially in their revision of the criteria employed in the choice and determination of *postulates*, that is, in the choice of those propositions which in each separate branch of deductive science are to be admitted without demonstration. Instead of conceiving of the difference between postulates and the other propositions which are demonstrated by means of them as consisting in the possession on the part of the former of some special character which renders them *per se* more acceptable, more evident, less disputable, and so on; the mathematical logicians regard postulates as propositions *on a par with all the others*. The choice of such "postulates" may differ according to the *end* in view, and must, in any case, depend upon an examination of the relations of dependence or connection which may be established between these "postulates" and the remaining propositions of a given theory, and upon a comparison with the form into which the treatment as a whole would develop under conditions of varying choices. If the relations between postulates and dependent propositions might formerly have been likened to those which subsist, in a state under autocratic *régime*, between the monarch or the privileged class and the rest of the social body, the work of the mathematical logicians

has been somewhat similar to that of the inaugurators of a constitutional or democratic *régime*, under which the choice of the rulers depends, at least theoretically, upon their recognized ability to exercise temporarily determined functions, to the public welfare. Postulates have had to relinquish that species of "divine right" with which their pretended evidence seemed to invest them, and resign themselves to becoming, instead of the arbiters, the *servi servorum*—the employees simply—of the great "associations" of propositions which make up the various branches of mathematics. With this same tendency are connected also the requirements regarding the maximum reduction of their number and pregnancy, the exact determination of their applicabilities and spheres of validity, and so forth.

* * *

A second resemblance, of not less importance, between the pragmatists and the mathematical logicians appears in their common repugnance to the vague, indefinite, generic; and in their care to analyze every assertion into its simplest terms, whether referring directly to *facts*, or to *connections between facts*. It is thus that both constituencies have come, each for itself and in its own way, to recognize the unreality of a great part of the distinctions which have been handed down from scholastic logic to the modern "theories of knowledge," and to subject others of these distinctions to critical analyses from which they have emerged in a sense transfigured, restored, enriched with new and more important significances.

Thus, by introducing the conception of "possible definition" a clear recognition has been reached of the wholly relative character of the distinction between the "essential properties" of a given figure or mathematical entity and the other properties possessed by it. In like manner the distinction between affirmative propositions and negative, and that between particular propositions and

general, have been absorbed in the one and more important distinction between propositions affirming the interdependence of two facts and propositions affirming the *possibility* or the *"non-absurdity"* of the coexistence of two or more facts. This involves, too, the invalidating of the distinction between categorical and hypothetical general propositions.

The recognition of the hypothetical nature of general propositions has also helped to turn attention to the "tacit restrictions," or unspecified limitations, upon which their validity depends. A good example of this is the observation of Maxwell (cited by Roiti in his *Elementi di Fisica*, 1894, p. 65): that even the simplest propositions as to areas, *e. g.*, that "the area of a triangle is given by one-half the product of the base by the altitude," would cease to be true if, instead of taking as the unit of measure of areas the square with side of unit length, one were to take the triangle having such unit as base and altitude. These considerations are intimately connected with those by which the pragmatists have been led to a more precise determination of the difference expressed in ordinary speech by opposing "laws" to facts; and to put on an entirely new basis the classic controversy between determinists and freedomists. (See *Leonardo*, April, 1905, p. 57, and Poincaré, *Valeur de la science.*)

* * *

A third point of contact between pragmatists and mathematical logicians consists in the interest shown on both sides for historical researches in the development of scientific theories. Both consider these essential as a means of recognizing the equivalence or identity of theories, under the various forms which they have assumed in different epochs or fields—all the time expressing substantially the same facts and serving the same ends. The logicians as well as the pragmatists have thus contributed to destroy

a number of prejudices attributed to supposed incompatibilities between the theories now current and the views of the great scientists or thinkers of antiquity. The fact has been brought to light that many, and those not the least important, among the discoveries of modern mathematicians, have consisted in nothing more than the introduction of new methods or notation for processes formerly employed or considered under other names by their predecessors. In the *Formulario* of Peano the importance given to historical data has steadily increased, especially under the inspiration of one of the principal collaborators. Vacca (among other things an enthusiastic investigator of the development of mathematics in the Far East); and the importance attributed to articles of this kind now constitutes one of the most noteworthy among the distinctive characteristics of the method of treatment of the various branches of mathematics that the said *Formulario* presents. Theories are therein expounded, not as in the ordinary treatment, under their "static" aspect—as one might express it,—their aspect of repose; but under that of movement and development—not in the conventional attitudes of stuffed animals, with glass eyes; but as organisms, which live, eat, struggle, reproduce: or at least like figures in a cinematograph, with some naturalness of progression and development.

To this tendency to recognize the identity of theories beyond or under differences of expression, symbolism, language, representative conventions and the rest, is to be attributed also the constant interest of the mathematical logicians in linguistic questions—from Grassmann, at once the author of the *Ausdehnungslehre* and of the *Wörterbuch zum Rig-Veda*, to Nagy, student of the transmission of Greek thought through the Syriac and Arabic commentaries; from Couturat, joint author with Leau of a History of the Projects of "Universal Language," to

Peano, inventor and propagandist of one of the most practical among them: the *"latino non flexo."*

* * *

Quite a different series of relations between pragmatists and mathematical logicians is offered by the important progress made by the latter in the theory of "definition."

First of all, the traditional method which makes definitions consist in the search for *genera* and *specific differences*, i. e., in the search for classes from which the class to be defined may result through the mediation of a "logical product," has been broadened so as to include every case in which the class to be defined may be obtained *as a function* of known classes, by means of any previously admitted operation or series of operations.

In another direction the scholastic methods of definition have been broadened by taking into consideration the cases in which that which is defined is not an isolated word but a group of words or phrases in which this word appears (*implicit definitions*). Hereby we have come to recognize more clearly than did, say, Aristotle, that definitions of isolated words are only a particular case, the simplest, in the vaster field of "implicit definitions." We see, for example, that to define a noun A signifies nothing more nor less than to indicate the sense which would be attributed to the phrase: "this thing or that is an A." Moreover it has become possible to characterize and justify the procedure, already instinctively followed by mathematicians, of employing successively diverse definitions of one and the same notation, according to the fields (whether inclusive or not) in which arises the opportunity of making use of groups of symbols in which this notation figures (definitions preceded by hypotheses limiting, and varying with, the variation of the definitions).

Particular interest, in their relations to pragmatism, is presented by what were called (Peano) *"definitions by abstraction"*; in which, from the fact that a given relation presents some of the characteristic properties of equality, occasion is taken to fashion a new concept: as, for example, from the fact that two straight lines parallel to a third are parallel to each other, is drawn the concept "direction"; or, from the fact that two amounts of merchandise exchanged for one and the same amount of a third commodity are mutually exchangeable, is evolved the concept "value."

A character common to the latter and to the other above-mentioned innovation, among those introduced by the mathematical logicians into the traditional theory of definition, consists in their tendency to bring to light the various orders of circumstances upon which may depend the fact that of a given word, taken by itself, a definition in the ordinary sense cannot be given; that is, a phrase cannot be enunciated which will indicate directly the characteristic or characteristics belonging to the objects to which the word in question is applied.

Not only has mathematical logic led to a recognition of the fact that to speak of the "definability" or "non-definability" of a given word or concept is to use a meaningless phrase, so long as no precise indication is given as to what *other words* or concepts may be used in the desired definition; but it has also afforded an explanation of the fact that many among the most important words of science and philosophy are found to be among those very ones of which it is unreasonable to ask or to seek a definition, in the scholastic sense. Mathematical logic has thus contributed most efficaciously to the defense of the position of the pragmatists against the "agnostic" prejudice which attributes the impossibility of the resolution of such problems to a pretended incapacity of the human mind to penetrate the "essence" of things.

The so-called "definitions by postulates"—i. e., those

which consist in determining the significance of a sign of operation or of relation by enunciating a certain number of norms which, by hypothesis, are to regulate its application—have, on the other hand, affinities with pragmatism in that they conduce to a clearer recognition of the arbitrary character of postulates, as well as of definitions. These appear in their true quality as propositions which possess the function of determining, in view of given ends or applications, the various fields of research; that is, as propositions whose sole justification consists in the importance and utility of the *consequences* which it may be possible to deduce therefrom.

* * *

Another characteristic of mathematical logic, in which, perhaps even more than in any of the afore-mentioned, is shown its affinity to pragmatism, is that which relates to the function which has come to be assumed by the search for and construction of "particular interpretations," or concrete examples, as criteria for determining the mutual independence, or the compatibility, of given affirmations or hypotheses.

Originally considered as simply a means of ascertaining the *necessity* (indispensability) of given premises, or the impossibility of reaching determinate conclusions without their aid, this search for particular examples has come to be regarded as the *only* process capable of guaranteeing that any group of hypotheses does not contain "implicit contradictions." That is, the construction of concrete interpretations, by which all the premises or hypotheses underlying a given deductive theory may be simultaneously verified, has assumed the importance of a condition in the absence of which even the most rigorous lines of reasoning can lead only to conclusions liable to contradiction by others, which may be obtained through deductions not less rigorous *than the premises themselves.*

Furthermore, in the choice of examples *hierarchies* have come to be formed, according to their degree of concreteness and determinateness. To such of these as are the most concrete and determinate of all—i. e., to the examples which belong to the field of arithmetic—has been attributed for the above-mentioned purpose, by some, a superiority over all others; particularly over those which imply considerations of continuity, or which belong to fields in which it is found more difficult to effect an exact and complete characterization or formulation of the facts adduced.

In this need of reinforcement by particular facts which is inherent in the more abstract theories (and this need is in direct proportion to their abstractness)—not, indeed, a need of facts which shall serve to confirm or to render inductively probable the separate premises upon which they are based; but of facts which shall guarantee the possibility of the *coexistence* and *cooperation* of such premises:—in this need of pure logic to derive strength, like Antæus, from periodic contact with the earth, one cannot fail to recognize one of the most significant indications of that mysterious alliance between "the extremes of theoretic activity"* (the intuition *of the particular* and the impulse to abstraction and generalization) which it is not the least of the merits of the pragmatic theories to have noted and proclaimed.*

* * *

Pragmatists and mathematicians find themselves in agreement, too, in their efforts toward the maximum of *conciseness* and *rapidity* of expression—in their tendency to eliminate all superfluity and redundance both of wording and of concept.

Both find the value of theories and doctrines not only in that which is said but also in what is *unmentioned* and

* Cf. G. Papini, "Les extrêmes de l'activité théorique" (in *Comptes rendus du IIe congrès internationale de philosophie.*) Geneva: Kündig, 1905.

whereof exposition or consideration is suppressed. See the article of Giuliano il Sofista on the Nourishment of Fasting, (*Leonardo,* April, 1905).

One of the principal achievements of mathematical logic consists in this very recognition of the fact that so many of what pass for *mathematical truths* owe their existence solely to imperfections of notation, which permit the enunciation of the same fact in different ways—to have afterward the pleasure of recognizing it as one and the same under its divers expressions. An example of this is to be found in the propositions of trigonometry, which re-enunciate, in new garb, theorems of elementary geometry; and, furthermore, re-enunciate them in manifold forms, of which the trigonometrical identities do no more than to express the equivalence.

By the introduction of other new symbols the number of "truths" of this kind might be increased indefinitely—repeating in science the miracle of the multiplication of the loaves and fishes; with the difference only that the results thus obtained would serve much more to distend than to nourish the minds to which they should be communicated. Indeed in this connection, as my friend G. Vacca observes, one might enunciate a law in form analogous to Malthus's Law, consisting in this: that when the concepts or the words which are introduced into a theory increase in arithmetical ratio, the corresponding propositions—whose truth or falsity the "science," to be complete, must needs decide—increase more rapidly than any geometrical progression (following an exponential law, enunciated by Clifford. See Peano, *Calcolo geometrico*, 1888).

Against a similar fatty degeneration of theories pragmatism, likewise, represents an energetic reaction; insisting as it does on the *instrumental* character of theories—affirming that they are not an *end in themselves,* but *media* and "organisms" whose efficacy and value is rigorously

dependent upon their agility, upon the absence of encumbrances and hindrances to their movements, upon their resemblance rather to lions and tigers than to hippopotami and mastodons. The favorite dictum of Plato: *ρεῖττον ἥμισυ παντός* is no less applicable to scientific theories than to any other branch of human activity.

GIOVANNI VAILATI