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Five hundred and eighty-fifth Meeting.

September 10, 1867. — ADJOURNED STATUTE MEETING.

The PRESIDENT in the chair.

The President called the attention of the Academy to the recent decease of Dr. J. Mason Warren and Dr. James Jackson, of the Resident Fellows; of Jeremiah Day, former President of Yale College, of the Associate Fellows; and of Sir William Lawrence, Augustus Boeckh, and Michael Faraday, of the Foreign Honorary Members.

The following paper was presented: —

Upon the Logic of Mathematics. By C. S. PEIRCE.

PART I.

THE object of the present paper is to show that there are certain general propositions from which the truths of mathematics follow syllogistically, and that these propositions may be taken as definitions of the objects under the consideration of the mathematician without involving any assumption in reference to experience or intuition. That there actually are such objects in experience or pure intuition is not in itself a part of pure mathematics.

Let us first turn our attention to the logical calculus of Boole. I have shown in a previous communication to the Academy, that this calculus involves eight operations, viz. Logical Addition, Arithmetical Addition, Logical Multiplication, Arithmetical Multiplication, and the processes inverse to these.

Definitions.

1. *Identity.* $a =_{,} b$ expresses the two facts that any a is b and any b is a.
2. *Logical Addition.* $a +_{,} b$ denotes a member of the class which contains under it all the a's and all the b's, and nothing else.
3. *Logical Multiplication.* a, b denotes only whatever is both a and b.
4. *Zero* denotes *nothing*, or the class without extent, by which we mean that if a is any member of any class, $a +_{,} 0$ is a.
5. *Unity*, denotes *being*, or the class without content, by which we mean that, if a is a member of any class, a is $a, 1$.
6. *Arithmetical Addition.* $a + b$, if $a, b =_{,} 0$ is the same as $a +_{,} b$, but, if a and b are classes which have any extent in common, it is not a class.

7. *Arithmetical Multiplication.* $a\,b$ represents an event when a and b are events only if these events are independent of each other, in which case $a\,b =_{,} a, b$. By the events being independent is meant that it is possible to take two series of terms, A_1, A_2, A_3, &c., and B_1, B_2, B_3, &c., such that the following conditions will be satisfied. (Here x denotes any individual or class, not nothing; A_m, A_n, B_m, B_n, any members of the two series of terms, and ΣA, ΣB, $\Sigma (A, B)$ logical sums of some of the A_n's, the B_n's, and the (A_n, B_n)'s respectively).

Condition	1.	No A_m is A_n.
"	2.	No B_m is B_n.
"	3.	$x =_{,} \Sigma (A, B)$
"	4.	$a =_{,} \Sigma A$.
"	5.	$b =_{,} \Sigma B$.
"	6.	Some A_m is B_n.

From these definitions a series of theorems follow syllogistically, the proofs of most of which are omitted on account of their ease and want of interest.

Theorems.

I.

If $a =_{,} b$, then $b =_{,} a$.

II.

If $a =_{,} b$, and $b =_{,} c$, then $a =_{,} c$.

III.

If $a +_{,} b =_{,} c$, then $b +_{,} a =_{,} c$.

IV.

If $a +_{,} b =_{,} m$ and $b +_{,} c =_{,} n$ and $a +_{,} n =_{,} x$, then $m +_{,} c =_{,} x$.

Corollary. — These last two theorems hold good also for arithmetical addition.

V.

If $a + b =_{,} c$ and $a' + b =_{,} c$, then $a =_{,} a'$, or else there is nothing not b.

This theorem does not hold with logical addition. But from definition 6 it follows that

No a is b (supposing there is any a)

No a' is b (supposing there is any a')

neither of which propositions would be implied in the corresponding formulæ of logical addition. Now from definitions 2 and 6,

Any a is c

$\therefore$ Any a is c not b

But again from definitions 2 and 6 we have

Any c not b is a' (if there is any not b)

$\therefore$ Any a is a' (if there is any not b)

And in a similar way it could be shown that any a' is a (under the same supposition). Hence by definition 1,

$a =_{,} a'$ if there is anything not b.

Scholium. — In arithmetic this proposition is limited by the supposition that b is finite. The supposition here though similar to that is not quite the same.

VI.

If $a, b =_{,} c$, then $b, a =_{,} c$.

VII.

If $a, b =_{,} m$ and $b, c =_{,} n$ and $a, n =_{,} x$, then $m, c =_{,} x$.

VIII.

If $m, n =_{,} b$ and $a +_{,} m =_{,} u$ and $a +_{,} n =_{,} v$ and $a +_{,} b =_{,} x$, then $u, v =_{,} x$.

IX.

If $m +_{,} n =_{,} b$ and $a, m =_{,} u$ and $a, n =_{,} v$ and $a, b =_{,} x$, then $u +_{,} v =_{,} x$.

The proof of this theorem may be given as an example of the proofs of the rest.

It is required then (by definition 3) to prove three propositions, viz.

1st. That any u is x.

2d. That any v is x.

3d. That any x not u is v.

First Proposition.

Since $u =_{,} a, m$ by definition 3

Any u is m,

and since $m +_{,} n =_{,} b$ by definition 2

Any m is b,

whence Any u is b,

But since $u =_{,} a, m$ by definition 3

Any u is a,

whence Any u is both a and b,

But since $a, b =_{,} x$ by definition 3

Whatever is both a and b is x

whence Any u is x.

Second Proposition.

This is proved like the first.

Third Proposition.

Since $a, m =_{,} u$ by definition 3,

Whatever is both a and m is u.

or Whatever is not u is not both a and m.

or Whatever is not u is either not a or not m.

or Whatever is not u and is a is not m.

But since $a, b =_{,} x$ by definition 3

Any x is a,

whence Any x not u is not u and is a,

whence Any x not u is not m.

But since $a, b =_{,} x$ by definition 3

Any x is b,

whence Any x not u is b,

Any x not u is b not m.

But since $m +_{,} n =_{,} b$ by definition 2

Any b not m is n,

whence Any x not u is n,

and therefore Any x not u is both a and m.

But since $a, n =_{,} v$ by definition 3

Whatever is both a and u is v,

whence Any x not u is v.

Corollary 1. — This proposition readily extends itself to arithmetical addition.

Corollary 2. — The converse propositions produced by transposing the last two identities of Theorems VIII. and IX. are also true.

Corollary 3. — Theorems VI., VII., and IX. hold also with arithmetical multiplication. This is sufficiently evident in the case of theorem VI., because by definition 7 we have an additional premise, namely, that a and b are independent, and an additional conclusion which is the same as that premise.

In order to show the extension of the other theorems, I shall begin with the following lemma. If a and b are independent, then corresponding to every pair of individuals, one of which is both a and b, there is just one pair of individuals one of which is a and the other b; and conversely, if the pairs of individuals so correspond, a and b are independent. For, suppose a and b independent, then, by definition 7, condition 3, every class (A_m, B_n) is an individual. If then A_a denotes any

A_m which is a, and B_b any B_m which is b, by condition 6 (A_a, B_n) and (A_m, B_b) both exist, and by conditions 4 and 5 the former is any individual a, and the latter any individual b. But given this pair of individuals, both of the pair (A_a, B_b) and (A_m, B_n) exist by condition 6. But one individual of this pair is both a and b. Hence the pairs correspond, as stated above. Next, suppose a and b to be any two classes. Let the series of A_m's be a and not-a; and let the series of B_m's be all individuals separately. Then the first five conditions can always be satisfied. Let us suppose, then, that the sixth alone cannot be satisfied. Then A_p and B_q may be taken such that (A_p, B_q) is nothing. Since A_p and B_q are supposed both to exist, there must be two individuals (A_p, B_n) and (A_m, B_q) which exist. But there is no corresponding pair (A_m, B_n) and (A_p, B_q). Hence, no case in which the sixth condition cannot be satisfied simultaneously with the first five is a case in which the pairs rightly correspond; or, in other words, every case in which the pairs correspond rightly is a case in which the sixth condition can be satisfied, provided the first five can be satisfied. But the first five can always be satisfied. Hence, if the pairs correspond as stated, the classes are independent.

In order to show that Theorem VII. may be extended to arithmetical multiplication, we have to prove that if a and b, b and c, and a and (b,c), are independent, then (a, b) and c are independent. Let s denote any individual. Corresponding to every s with (a, b, c), there is an a and (b, c). Hence, corresponding to every s with s and with (a, b, c) (which is a particular case of that pair), there is an s with a and with (b, c). But for every s with (b, c) there is a b with c; hence, corresponding to every a with s and with (b, c), there is an a with b and with c. Hence, for every s with s and with (a, b, c) there is an a with b and with c. For every a with b there is an s with (a, b); hence, for every a with b and with c, there is an s with (a, b) and c. Hence, for every s with s and with (a, b, c) there is an s with (a, b) and with c. Hence, for every s with (a, b, c) there is an (a, b) with c. The converse could be proved in the same way. Hence, &c.

Theorem IX. holds with arithmetical addition of whichever sort the multiplication is. For we have the additional premise that "No m is n"; whence since "any u is m" and "any v is n," "no u is v," which is the additional conclusion.

Corollary 2, so far as it relates to Theorem IX., holds with arithmetical addition and multiplication. For, since no m is n, every pair, one of which is a and either m or n, is either a pair, one of which is a and m, or a pair, one of which is a and n, and is not both. Hence, since for every pair one of which is a and m, there is a pair one of which is a and the other m, and since for every pair one of which is a, n there is a pair one of which is a and the other n; for every pair one of which is a and either m or n, there is either a pair one of which is a and the other m, or a pair one of which is a and the other n, and not both; or, in other words, there is a pair one of which is a and the other either m or n.

[It would perhaps have been better to give this complicated proof in its full syllogistic form. But as my principal object is merely to show that the various theorems could be so proved, and as there can be little doubt that if this is true of those which relate to arithmetical addition it is true also of those which relate to arithmetical multiplication, I have thought the above proof (which is quite apodeictic) to be sufficient. The reader should be careful not to confound a proof which needs itself to be experienced with one which requires experience of the object of proof.]

X.

If $a\, b =_{,} c$ and $a'\, b =_{,} c$, then $a =_{,} a'$, or no b exists.

This does not hold with logical, but does with arithmetical multiplication.

For if a is not identical with a', it may be divided thus

$$a =_{,} a\,,\, a' + a\,,\, \bar{a}'$$

if $\bar{a}'$ denotes not a'. Then

$$a\,,\, b =_{,} (a\,,\, a')\,,\, b + (a\,,\, \bar{a}')\,,\, b$$

and by the definition of independence the last term does not vanish unless $(a\,,\, \bar{a}') =_{,} 0$ or all a is a'; but since $a\,,\, b =_{,} a'\,,\, b =_{,} (a\,,\, a')\,,\, b + (\bar{a}\,,\, a')\,,\, b$, this term does vanish, and, therefore, only a is a', and in a similar way it could be shown that only a' is a.

XI.

$1 +_{,} a =_{,} 1.$

This is not true of arithmetical addition, for since by definition 7,

$$1\, x\,,\, 1 =_{,} x\, 1$$

by Theorem IX.

$$x\,,\, (1 + a) =_{,} x\,(1 + a) =_{,} x\, 1 + x.a =_{,} x + x\, a$$

Whence $x\, a =_{,} 0$, while neither x nor a is zero, which, as will appear directly, is impossible.

XII.

$0\,,\, a =_{,} 0$

Proof. — For call $0\,,\, a =_{,} x$. Then by definition 3

x belongs to the class *zero.*

∴ by definition 4 $\qquad x =_{,} 0.$

Corollary 1. — The same reasoning applies to arithmetical multiplication.

Corollary 2. — From Theorem X. and the last corollary it follows that if $a\, b =_{,} 0$, either $a =_{,} 0$ or $b =_{,} 0$.

XIII.

$a\,,\, a =_{,} a.$

XIV.

$a +_{,} a =_{,} a.$

These do not hold with arithmetical operations.

General Scholium. — This concludes the theorems relating to the direct operations. As the inverse operations have no peculiar logical interest, they are passed over here.

In order to prevent misapprehension, I will remark that I do not undertake to demonstrate the principles of logic themselves. Indeed, as I have shown in a previous paper, these principles considered as speculative truths are absolutely empty and indistinguishable. But what has been proved is the *maxims* of logical procedure, a certain system of signs being given.

The definitions given above for the processes which I have termed arithmetical plainly leave the functions of these operations in many cases uninterpreted. Thus if we write

$$a + b \underset{,}{=} b + a$$

$$a + (b + c) \underset{,}{=} (a + b) + c$$

$$b\, c \underset{,}{=} c\, b$$

$$(a\, b)\, c \underset{,}{=} a\, (b\, c)$$

$$a\, (m + n) \underset{,}{=} a\, m + a\, n$$

we have a series of identities whose truth or falsity is entirely undeterminable. In order, therefore, *fully to define those operations*, we will say that all propositions, equations, and identities which are in the general case left by the former definitions undetermined as to truth shall be true, provided they are so in all interpretable cases.

On Arithmetic.

Equality is a relation of which identity is a species.

If we were to leave equality without further defining it, then by the last scholium all the formal rules of arithmetic would follow from it. And this completes the central design of this paper, as far as arithmetic is concerned.

Still it may be well to consider the matter a little further. Imagine, then, a particular case under Boole's calculus, in which the letters are no longer terms of first intention, but terms of second intention, and that of a special kind. Genus, species, difference, property, and accident, are the well-known terms of second intention. These relate particularly to the *comprehension* of first intentions; that is, they refer to different sorts of predication. Genus and species, however, have at least a secondary reference to the *extension* of first intentions. Now let the letters, in the particular application of Boole's calculus now supposed, be terms of second intention which relate exclusively to the extension of first intentions. Let the differences of the characters of things and events be disregarded, and let the letters signify only the differences of classes as wider or narrower. In other words, the only logical comprehension which the letters considered as terms will have is the greater or less divisibility of the classes. Thus, n in another case of Boole's calculus might, for example, denote "New England State"; but in the case now supposed, all the characters which make these States what they are being neglected, it would signify only what

essentially belongs to a class which has the same relations to higher and lower classes which the class of New England States has, — that is, a collection of *six*.

In this case, the sign of identity will receive a special meaning. For, if m denotes what essentially belongs to a class of the rank of "sides of a cube," then $m \underset{,}{=} n$ will imply, not that every New England State is a side of a cube, and conversely, but that whatever essentially belongs to a class of the numerical rank of "New England States" essentially belongs to a class of the rank of "sides of a cube, and conversely. *Identity* of this particular sort may be termed *equality*, and be denoted by the sign $=$.* Moreover, since the numerical rank of a *logical sum* depends on the identity or diversity (in first intention) of the integrant parts, and since the numerical rank of a *logical product* depends on the identity or diversity (in first intention) of parts of the factors, logical addition and multiplication can have no place in this system. Arithmetical addition and multiplication, however, will not be destroyed. $a\, b = c$ will imply that whatever essentially belongs at once to a class of the rank of a, and to another independent class of the rank of b belongs essentially to a class of the rank of c, and conversely. $a + b = c$ implies that whatever belongs essentially to a class which is the logical sum of two mutually exclusive classes of the ranks of a and b belongs essentially to a class of the rank of c, and conversely. It is plain that from these definitions the same theorems follow as from those given above. *Zero* and *unity* will, as before, denote the classes which have respectively no extension and no comprehension; only the comprehension here spoken of is, of course, that comprehension which alone belongs to letters in the system now considered, that is, this or that degree of divisibility; and therefore *unity* will be what belongs essentially to a class of any rank independent of its divisibility. These two classes alone are common to the two systems, because the first intentions of these alone determine, and are determined by, their second intentions. Finally, the laws of the Boolian

* Thus, in one point of view, *identity* is a species of *equality*, and, in another, the reverse is the case. This is because the Being of the copula may be considered on the one hand (with De Morgan) as a special description of "inconvertible, transitive relation," while, on the other hand, all relation may be considered as a special determination of being. If a Hegelian should be disposed to see a contradiction here, an accurate analysis of the matter will show him that it is only a verbal one.

calculus, in its ordinary form, are identical with those of this other so far as the latter apply to *zero* and *unity*, because every class, in its first intention, is either without any extension (that is, is nothing), or belongs essentially to that rank to which every class belongs, whether divisible or not.

These considerations, together with those advanced on page 293 (§ 12) of this volume, will, I hope, put the relations of logic and arithmetic in a somewhat clearer light than heretofore.

Five hundred and eighty-sixth Meeting.

October 8, 1867. — MONTHLY MEETING.

The CORRESPONDING SECRETARY in the chair.

The Corresponding Secretary read letters relative to exchanges; also a letter from Major-General Sabine in acknowledgment of his election as Foreign Honorary Member of the Academy.

The Corresponding Secretary announced the recent decease of Hon. Charles G. Loring, of the Resident Fellows.

Dr. C. G. Putnam presented the meteorological observations of the late Dr. Jackson.

Professor Lovering presented for Professor Treadwell the following paper: —

Corrections to a Paper "On the Comparative Strength of Cannon of Modern Construction," published in Vol. VII. of the Proceedings of the Academy. By DANIEL TREADWELL.

IN a paper "On the Comparative Strength of Cannon of Modern Construction," written by me in January, 1866, communicated to the Academy in September of the same year, and published in the last volume (the seventh) of our Proceedings, I, by some inadvertence for which I am now unable to account, in computing the force of the 600 pounder, or 13·3-inch coil gun, as constructed by Armstrong, described it as capable of bearing a charge of 100 pounds of powder.

Although this quantity of powder was no doubt fired in it, I know not how many times, yet it ought not by any means to be rated as its *service charge;* and I recognize it as an oversight in me to have taken

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