PROCEEDINGS

OF THE

AMERICAN ACADEMY

OF

ARTS AND SCIENCES.

NEW SERIES.
~~Vol. II.~~

WHOLE SERIES.
VOL. X.

FROM MAY, 1874, TO MAY, 1875.

SELECTED FROM THE RECORDS.

BOSTON:
PRESS OF JOHN WILSON AND SON.
1875.

XIV.

ON THE APPLICATION OF LOGICAL ANALYSIS TO MULTIPLE ALGEBRA.

By C. S. Peirce.

Presented, May 11, 1875.

The letters of an algebra express the relation of the product to the multiplicand. Thus, iA expresses the quantity which is related to A in the manner denoted by i. This being the conception of these algebras, for each of them we may imagine another "absolute" algebra, as we may call it, which shall contain letters which can only be products and multiplicands, not multipliers. Let the general expression of the absolute algebra be $aI + bJ + cK + dL +$ etc. Multiply this by any letter i of the relative algebra, and denote the product by

$$(A_1a + A_2b + A_3c + \text{etc.})\ I.$$
$$+ (B_1a + B_2b + B_3c + \text{etc.})\ J.$$
$$+ \text{etc.}$$

Now we may obviously enlarge the given relative algebra, so that

$$i = A_1 i_{11} + A_2 i_{12} + A_3 i_{13} + \text{etc.}$$
$$+ B_1 i_{21} + B_2 i_{22} + B_3 i_{23} + \text{etc.}$$
$$+ \text{etc.}$$

where $i_{11}\ i_{12}$ etc. are such that the product of either of them into any letter of the absolute algebra shall equal some letter of that algebra. That there is no self-contradiction involved in this supposition seems axiomatic.

In this way each letter of the given algebra is resolved into a sum of terms of the form $aA:B$, a being a scalar, and $A:B$ such that

$$(A:B)\ (B:C) = A:C.$$
$$(A:B)\ (C:D) = 0.$$

The actual resolution is usually performed with ease, but in some cases a good deal of ingenuity is required. I have not found the process facilitated by any general rules. I have actually resolved all the Double, Triple, and Quadruple algebras, and all the Quintuple ones, that appeared to present any difficulty. I give a few examples.

$bi_s.$

	i	j	k	l	m
i	j	0	i	0	0
j	0	0	0	0	0
k	$j+al$	0	0	0	$bj+cl$
l	0	0	0	0	0
m	$aj+bl$	0	$cj+dl$	0	l

$$i = cd'\ A{:}B + b'\ B{:}C + b'\ D{:}E.$$
$$j = b'cd'\ A{:}C.$$
$$k = cd'\ A{:}B + acd'\ D{:}B + b'cd\ D{:}F + cd'\ E{:}C + b'cd\ A{:}F.$$
$$l = b'cd'\ D{:}C.$$
$$m = a'cd'\ A{:}B + b'c'\ A{:}E + b\,c''\ D{:}B + b'd'\ D{:}E + b'cd'\ D{:}F$$
$$+ F{:}C.$$

$bd_j.$

	i	j	k	l	m
i	j	0	l	0	0
j	0	0	0	0	0
k	$j+rl$	0	$i+n$	0	$-j-rl$
l	0	0	j	0	0
m	$(r^2-1)j$	0	$-l$	0	$-rj$

$$i = A{:}D + D{:}F + B{:}E + C{:}F.$$
$$j = A{:}F.$$
$$k = rA{:}B + rB{:}C + D{:}E - \tfrac{1}{r}D{:}F + E{:}F.$$
$$l = A{:}E - \tfrac{1}{r}A{:}F + B{:}E.$$
$$m = r^2 A{:}C - A{:}D - B{:}E - C{:}F.$$

$bk_s.$

	i	j	k	l	m	n
i	i	j	k	l	m	
j	j	k				
k	k					
l	l	ak		k		
m						k
n	n					

$$i = A:A + B:B + C:C + D:D.$$
$$j = A:B + B:C.$$
$$k = A:C.$$
$$l = aA:B + A:D + D:C.$$
$$m = A:E.$$
$$n = E:C.$$

$br_s.$

	i	j	k	l	m
i	j				
j					
k	l		m		
l					
m					

$$i = A:B + B:C.$$
$$j = A:C.$$
$$k = D:B + D:E + E:F.$$
$$l = D:C.$$
$$m = D:F.$$

XV.

ON THE USES AND TRANSFORMATIONS OF LINEAR ALGEBRA.

By Benjamin Peirce.

Presented, May 11, 1875.

Some definite interpretation of a linear algebra would, at first sight, appear indispensable to its successful application. Whereas it is a singular fact, and one quite consonant with the principles of sound logic, that its first and general use is mostly to be expected from its want of significance. The interpretation is a trammel to the use. Symbols are essential to comprehensive argument. The familiar proposition that all A is B, and all B is C, and therefore all A is C, is contracted in its domain by the substitution of significant words for the symbolic letters. The A, B and C are subject to no limitation for the purposes and validity of the proposition; they may represent not merely the actual, but also the ideal, the impossible as well as the possible. In Algebra, likewise, the letters are symbols which, passed through a machinery of argument in accordance with given laws, are developed into symbolic results under the name of formulas. When the formulas admit of intelligible interpretation, they are accessions to knowledge; but independently of their interpretation they are invaluable as symbolical expressions of thought. But the most noted instance is the symbol, called the impossible or imaginary, known also as the square root of minus one, and which, from a shadow of meaning attached to it, may be more definitely distinguished as the symbol of *semi-inversion*. This symbol is restricted to a precise signification as the representative of perpendicularity in quaternions, and this wonderful algebra of space is intimately dependent upon the special use of the symbol for its symmetry, elegance, and power. The immortal author of quaternions has shown that there are other significations which may attach to the symbol in other cases. But the strongest use of the symbol is to be found in its magical power of doubling the actual

The Corresponding Secretary presented a paper, by Dr. A. A. Hayes, "On the Wide Diffusion of Vanadium, and its Association with Phosphorus in many Rocks."

The President announced the death of the Rev. James Walker, D.D., Resident Fellow of the Academy.

Six hundred and seventy-sixth Meeting.

January 27, 1875. — STATED MEETING.

The PRESIDENT in the chair.

The following gentlemen were elected into the Academy: —

Henry Barker Hill, of Cambridge, to be a Resident Fellow in Class I., Section 3.

James Bradstreet Greenough, of Cambridge, to be a Resident Fellow in Class III., Section 2.

William James, of Cambridge, to be a Resident Fellow in Class II., Section 3.

Joachim Barrande, of Prague, to be a Foreign Honorary Member in Class II., Section 1, in place of the late Élie de Beaumont.

Louis Adolphe Thiers, of Paris, to be a Foreign Honorary Member in Class III., Section 3, in place of the late F. P. G. Guizot, of Paris.

Jean Léon Gérôme, of Paris, to be a Foreign Honorary Member in Class III., Section 4, in place of the late W. von Kaulbach, of Munich.

Professor E. C. Pickering communicated the results of some experiments on the "Foci of Lenses Placed Obliquely," by himself and Dr. Charles H. Williams.

The President read a letter from the President of the Geographical Society of Paris, inviting the Academy to send a delegate to the International Congress of Geographical Sciences, to be held at Paris during the coming summer.

Six hundred and seventy-seventh Meeting.

February 9, 1875. — MONTHLY MEETING.

The PRESIDENT in the chair.

The President announced the death of Dr. Charles G. Putnam, Resident Fellow.

It was voted to refer to the Council the letter read at the last meeting from the President of the Geographical Society.

Professor N. S. Shaler presented a communication on the Freezing of Water in Bomb-shells.

Six hundred and seventy-eighth Meeting.

March 9, 1875. — MONTHLY MEETING.

The PRESIDENT in the chair.

The Corresponding Secretary read a letter from the Royal Institution of Civil Engineers of Ireland, inviting an exchange of publications; also, a letter from the Royal Academy of Brussels, inviting subscriptions to a monument in honor of Quetelet; also, letters from Messrs. Hill, James, and Greenough, acknowledging their election as Fellows of the Academy; also, letters from James Clerk Maxwell, of Cambridge, England, and Jean Léon Gérôme, of Paris, acknowledging their election as Foreign Honorary Members.

The President announced the death of Sir Charles Lyell, Foreign Honorary Member.

The President announced that the letter from the Geographical Society of Paris had been considered by the Council; and, in accordance with their recommendation, it was voted to accept the invitation of the Society, and Hon. R. C. Winthrop was appointed to represent the Academy at the ensuing Congress.

Mr. C. S. Peirce presented a paper on " Photometric Measurements of the Stars."

Professor John Trowbridge presented, by title, a paper on the " Effect of Heat upon the Magnetic Susceptibility of Soft Iron," by H. Amory and F. Minot.

Professor W. Watson presented a copy of his work on "Descriptive Geometry" to the Academy.

Prof. Asa Gray presented, by title, a paper, "A Conspectus of the North American Hydrophyllaceæ."

Mr. Sereno Watson presented, by title, a paper on the "Revision of the Genus *Ceanothus*, and Descriptions of New Plants, with a Synopsis of the Western Species of *Silene*."

Dr. W. G. Farlow presented, by title, a paper on the Algæ of the United States.

Six hundred and seventy-ninth Meeting.

April 13, 1875. — MONTHLY MEETING.

The PRESIDENT in the chair.

Professor Wolcott Gibbs communicated the following "Optical Notices:" —

1. On a New Optical Constant.

2. On the Determination of Indices of Refraction without the use of Divided Instruments.

3. On a New Optical Instrument.

Professor John Trowbridge read a paper, "On a New Form of Induction Coil."

Professor Trowbridge also communicated two papers from the Physical Laboratory of Harvard College, by undergraduates of the University: —

1. On the Effects of Armatures upon Electro-Magnets, by B. O. Peirce and E. B. Lefavour.

2. On the Time of Demagnetization of Soft Iron, by W. C. Hodgkins and J. H. Jennings.

The Recording Secretary presented, by title, a paper, "On the Light transmitted through one or more Plates of Glass," by W. W. Jacques.

Six hundred and eightieth Meeting.

May 11, 1875. — MONTHLY MEETING.

The PRESIDENT in the chair.

The Corresponding Secretary presented, in print, the Report of the Council for the past year, which is hereto appended. He also read a letter from M. Thiers, acknowledging his election as Foreign Honorary Member of the Academy. The following scientific communications were then made, but the first two were only read by title: —

On the Application of Logical Analysis to Multiple Algebra, by C. S. Peirce.

On the Uses and Transformations of Linear Algebra, by Benjamin Peirce.

On the Intensity of Twilight, by Charles H. Williams.

On the Light of the Sky, by W. O. Crosby.

On Light Absorbed by the Atmosphere of the Sun, by E. C. Pickering and D. P. Strange.

On Tests of Magneto-Electric Machines, by E. C. Pickering and D. P. Strange.

Answer to M. Jamin's Objections to Ampère's Theory, by William W. Jacques.

On Melanosiderite: a New Mineral Species. from Mineral Hill, Delaware County. Pennsylvania, by Josiah P. Cooke, Jr.

On Two New Vermiculites. with a Revision of the other Members of this Group of Minerals, by Josiah P. Cooke, Jr., and F. A. Gooch.

On a Possible Explanation of the Method employed by Nobert in Ruling his Test Plates, by William A. Rogers.

On the Solar Motion and Stellar Distances, second paper, by T. H. Safford.