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ESPOSIZIONE DEL METODO DEI MINIMI QUADRATI.

PER ANNIBALE FERRERO, *Tenente Colonnello di Stato Maggiore, ec. Firenze, 1876.*

BY CHARLES S. PEIRCE, *New York.*

RECENT discussions in this country, of the literature of the method of Least Squares, have passed by without mention the views of the accomplished chief of the geodetical division of the Italian Survey, as set forth in the work above cited, which was first published, in part, in 1871. The subject is here, for the first time, in my opinion, set upon its true and simple basis; at all events the view here taken is far more worthy of attention than most of the proposed proofs of the method.

Lieut. Col. Ferrero begins by considering the principles of the arithmetical mean. A quantity having been directly observed, a number of times, independently, and under like circumstances, the value which might be inferred from the observations is, in the first place, a symmetrical function of the observed quantities; for, if the observations are independent, the order of their occurrences is of no consequence, and the circumstances under which they are taken, differ in no assignable respect, except that of being taken at different times. In the second place, the value inferred must be such a function of the values observed, that when the latter are all equal, the former reduces to this common value. The author calls functions having these two properties, (1st, that of being symmetrical with respect to all the variables, and 2d, that of reducing to the common value of the variables when these are all equal,) *means*. There is a whole class of functions of this sort, such as the arithmetic mean, the geometrical mean, the arithmetic-geometrical mean of Gauss, the quadratic mean,* and many others instanced in the text. It is shown, without difficulty, that these means are continuous functions, and that their value is intermediate between the extreme values of the different variables, when the latter do not differ greatly.

Let o', o'', o''', etc. denote the values given by the observations. Let n denote the number of the observations; let p denote the arithmetical mean;

* This seems the appropriate name for $\sqrt{\frac{[x^2]}{n}}$.

and let x', x'', x''', etc. denote the excess of the observed values over the a metical mean. Then write

$$V = f(o', o'', o''', \text{etc.})$$

for any mean of the observations. Develop this function according to po of x', x'', x''', etc. We have

$$V = f(p + x', p + x'', p + x''', \text{etc.})$$

$$= f(p, p, p, \text{etc.}) + \frac{dV}{dp}(x' + x'' + x''' + \text{etc.}) + \Delta;$$

where Δ denotes the terms of higher orders.

Since

$$x' + x'' + x''' + \text{etc.} = 0,$$

and

$$f(p, p, p, \text{etc.}) = p,$$

this reduces to

$$V = p + \Delta.$$

In considering the value of Δ, we may limit ourselves to terms of the se order. As the partial differentials of any species and order, relative o', o'', o''', etc. all become equal when x', x'', x''', etc. vanish, we may write

$$\frac{d^2V}{do'^2} = \frac{d^2V}{do''^2} = \frac{d^2V}{do'''^2} = \text{etc.} = \beta$$

$$\frac{d^2V}{do'.do''} = \frac{d^2V}{do''.do'''} = \text{etc.} = \gamma$$

then

$$\Delta = \tfrac{1}{2}\beta(x'^2 + x''^2 + x'''^2 + \text{etc.}) + \gamma(x'x'' + x'x''' + \text{etc.}).$$

But the square of $[x] = 0$ gives

$$\Sigma x x' = -\tfrac{1}{2}[x^2],$$

so that

$$\Delta = \frac{\beta - \gamma}{2}[x^2] = k\frac{[x^2]}{n},$$

where k is a quantity which does not increase indefinitely with n. when the observations are good, $\frac{[x^2]}{n}$ is not large, and, therefore, in su case no mean will differ very much from the arithmetical mean. The la being the simplest to deal with, may therefore be used without great di vantage. Such is, according to Colonel Ferrero, the utmost defence o principle which can be made to cover all the cases in which it is usu employ the method; and all further defence of it is more or less limited application.

In very many cases, however, it is easy to see that either in regard to the quantity directly observed, or in regard to some function of it, the zero of the scale of measurement, and the unit of the same scale, are both arbitrary. For instance, in photometric observations, this is true of the logarithm of the light. In such cases, considering such function to be the observed quantity, we have there two principles, first proposed, in connection with a really superfluous third one, by Schiaparelli.

1st. The mean to be adopted must be such that if each observed value is multiplied by any constant, the result is increased in the same ratio.

2d. The mean to be adopted must be one which is increased by a constant o, when each observed value is increased by the same constant.

Our author's treatment of these principles is exceedingly neat. Using the same notation as above, write

$$V = p + A_2 + A_3 \ldots + A_n \ldots$$

where A_n is the sum of the terms of the order n in x', x'', x''', etc. The general term A_n is, therefore, of the form $A_n = \alpha\Sigma x^n + \beta\Sigma x^{n-1}x' + \gamma\Sigma x^{n-2}x''^2 \ldots + \zeta\Sigma x'^r x''^s x'''^t \ldots$ where Σ expresses the symmetrical sum of similar terms. In the general term $r + s + t + \text{etc.} = n$. Since ζ is evidently a function of p, we may put $\zeta = \phi(p)$, and it remains to find the form of this function. Multiplying every o by c, p is changed to cp, x to cx, and the general term $\zeta\Sigma x'^r x''^s x'''^t$ etc. $= \phi(p)\Sigma x'^r x''^s x'''^t$ etc. is changed to $\phi(cp)c^n\Sigma x'^r x''^s x'''^t$ etc. Since, therefore, V is changed to cV, we have $\phi(cp)c^n = \phi(p)c$. Putting $p = 1$, $\phi(c) = \frac{\phi(1)}{c^{n-1}}$. Denoting this numerator by ξ_1, the general term becomes

$$A_n = \frac{1}{p^{n-1}}[\alpha_1\Sigma x^n + \ldots + \xi_1\Sigma x'^r x''^s x'''^t \ldots + \ldots],$$

where α, ξ, etc., are numerical coefficients independent of p. From this circumstance it follows that the quantity in square brackets, which may be called A'_n, does not change when the same constant quantity k is added to all the observed quantities o', o'', o''', etc.; for such an addition only increases p by this same constant, and leaves x', x'', x''', etc., unchanged. Thus the mean in question, which may now be written

$$V = p + \frac{A'_2}{p} + \frac{A'_3}{p^2} + \text{etc.},$$

becomes, in consequence of such an addition,

$$V_k = p + k + \frac{A'_2}{p + k} + \frac{A'_3}{(p + k)^2} + \text{etc.}$$

But by principle No. 2, it becomes,

$$V_k = p + k + \frac{A'_2}{p} + \frac{A'_3}{p^2} = \text{etc.}$$

So that, $A'_2 = A'_3 = \text{etc.} = 0$, and we have

$$V = p,$$

or the arithmetical mean is the only one which conforms to the conditions.

Another still more special case, is that contemplated by the de strations of Laplace, Poisson, Hagen, Crofton, etc. It is treated b author, but need not be considered in this notice.

It may be of interest to see how Colonel Ferrero is able, without l least squares expressly upon the theory of probabilities, to derive th mula for finding mean error. Using always the same notation, he term

$$m = \sqrt{\frac{[x^2]}{n}}$$

the *mean residual* of the observations.

Suppose, then, that there be an indefinitely great *series of series* of vations of the same quantity, each lesser series consisting of n observ and each having the same mean residual. Then, there being an i number of such series, the mean of their mean results may be taken true value, by definition. For the ultimate result of indefinitely cont observation is all that we aim at in sciences of observation. Then the ber of the lesser series being q, the result will be

$$V = \frac{[p]}{q}.$$

Adopt the notation

$$\delta = p - V \quad \delta_1 = p_1 - V \quad \delta_2 = p_2 - V, \text{ etc.},$$

then δ, δ_1, δ_2, etc., are the true errors of p, p_1, p_2, etc. Let y_0, y'_0, y''_0, etc. b true errors of the first series of observations, y_1, y'_1, y''_1 etc. those of the s series, and so for the others. We have, then, $y = o - V = o - p + \delta = $

Squaring and summing for the nq values of y, we have

$$\Sigma y^2 = \Sigma x^2 + \Sigma\delta^2 + 2\Sigma x\Sigma\delta$$

or, since $\Sigma x = 0$, and $\Sigma\delta = 0$,

$$\Sigma y^2 = \Sigma x^2 + \Sigma\delta^2.$$

Now if η be the quadratic mean of the error of p, we have $\Sigma\delta^2 = nq$

$$\Sigma y^2 = nqm^2 + nq\eta^2,$$

or the mean error μ of an observation is given by

$$\mu^2 = \frac{\Sigma y^2}{nq} = m^2 + \eta^2.$$

But it is easily shown (from the equality of positive and negative errors) that

$$\eta^2 = \frac{\mu^2}{n}$$

whence

$$\mu = \sqrt{\frac{[x^2]}{n-1}}.$$

With regard to the mode of passing from the principle of the arithmetical mean to the general method of least squares, the best way seems to be first to prove that the solution of the equations

$$a_1x = n_1$$
$$a_2x = n_2$$
$$\text{etc.},$$

is $x = \dfrac{[an]}{[a^2]}$. This is easy, after the rule for the error of a mean is established. Then, having given the equations

$$a_1x + b_1y + c_1z + \text{etc.} = n_1$$
$$a_2x + b_2y + c_2z + \text{etc.} = n_2;$$

first, consider these as similar to the equations just given; thus,

$$a_1x = n_1 - b_1y - c_1z - \text{etc.},$$
$$a_2x = n_2 - b_2y - c_2z - \text{etc.},$$
$$\text{etc.},$$

whence we obtain the first normal equation,

$$x = \frac{[an_1] - [ab]\,y - [ac]\,z - \text{etc.}}{[a^2]}$$

and the others in a similar way.

The treatise of Colonel Ferrero may be recommended to those desirous of having a thorough practical acquaintance with the method, as decidedly the best and clearest on the subject.