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# *On the Logic of Number*

BY C. S. PEIRCE.

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Nobody can doubt the elementary propositions concerning number: those that are not at first sight manifestly true are rendered so by the usual demonstrations. But although we see they *are* true, we do not so easily see precisely *why* they are true; so that a renowned English logician has entertained a doubt as to whether they were true in all parts of the universe. The object of this paper is to show that they are strictly syllogistic consequences from a few primary propositions. The question of the logical origin of these latter, which I here regard as definitions, would require a separate discussion. In my proofs I am obliged to make use of the logic of relatives, in which the forms of inference are not, in a narrow sense, reducible to ordinary syllogism. They are, however, of that same nature, being merely syllogisms in which the objects spoken of are pairs or triplets. Their validity depends upon no conditions other than those of the validity of simple syllogism, unless it be that they suppose the existence of singulars, while syllogism does not.

The selection of propositions which I have proved will, I trust, be sufficient to show that all others might be proved with like methods.

Let $r$ be any relative term, so that one thing may be said to be $r$ of another, and the latter $r$'d by the former. If in a certain system of objects, whatever is $r$ of an $r$ of anything is itself $r$ of that thing, then $r$ is said to be a transitive relative in that system. (Such relatives as "lover of everything loved by —" are transitive relatives.) In a system in which $r$ is transitive, let the $q$'s of anything include that thing itself, and also every $r$ of it which is not $r$'d by it. Then $q$ may be called a fundamental relative of quantity; its properties being, first, that it is transitive; second, that everything in the system is $q$ of itself, and, third, that nothing is both $q$ of and $q$'d by anything except itself. The objects of a system having a fundamental relation of quantity are called quantities, and the system is called a system of quantity.

A system in which quantities may be $q$'s of or $q$'d by the same quantity without being either $q$'s of or $q$'d by each other is called multiple;* a system in which of every two quantities one is a $q$ of the other is termed simple.

*Simple Quantity.*

In a simple system every quantity is either "as great as" or "as small as" every other; whatever is as great as something as great as a third is itself as great as that third, and no quantity is at once as great as and as small as anything except itself.

A system of simple quantity is either continuous, discrete, or mixed. A continuous system is one in which every quantity greater than another is also greater than some intermediate quantity greater than that other. A discrete system is one in which every quantity greater than another is next greater than some quantity (that is, greater than without being greater than something greater than). A mixed system is one in which some quantities greater than others are next greater than some quantities, while some are continuously greater than some quantities.

*Discrete Quantity.*

A simple system of discrete quantity is either limited, semi-limited, or unlimited. A limited system is one which has an absolute maximum and an absolute minimum quantity; a semi-limited system has one (generally considered a minimum) without the other; an unlimited has neither.

A simple, discrete, system, unlimited in the direction of increase or decrement, is in that direction either infinite or super-infinite. An infinite system is one in which any quantity greater than $x$ can be reached from $x$ by successive steps to the next greater (or less) quantity than the one already arrived at. In other words, an infinite, discrete, simple, system is one in which, if the quantity next greater than an attained quantity is itself attained, then any quantity greater than an attained quantity is attained; and by the class of attained quantities is meant any class whatever which satisfies these conditions. So that we may say that an infinite class is one in which if it is true that every quantity next greater than a quantity of a given class itself belongs to that class, then it is true that every

* For example, in the ordinary algebra of imaginaries two quantities may both result from the addition of quantities of the form $a^2 + b^2i$ to the same quantity without either being in this relation to the other.



quantity greater than a quantity of that class belongs to that class. Let the class of numbers in question be the numbers of which a certain proposition holds true. Then, an infinite system may be defined as one in which from the fact that a certain proposition, if true of any number, is true of the next greater, it may be inferred that that proposition if true of any number is true of every greater.

Of a super-infinite system this proposition, in its numerous forms, is untrue.

*Semi-infinite Quantity.*

We now proceed to study the fundamental propositions of semi-infinite, discrete, and simple quantity, which is ordinary number.

*Definitions.*

The minimum number is called one.

By $x + y$ is meant, in case $x = 1$, the number next greater than $y$; and in other cases, the number next greater than $x' + y$, where $x'$ is the number next smaller than $x$.

By $x \times y$ is meant, in case $x = 1$, the number $y$, and in other cases $y + x'y$, where $x'$ is the number next smaller than $x$.

It may be remarked that the symbols $+$ and $\times$ are triple relatives, their two correlates being placed one before and the other after the symbols themselves.

*Theorems.*

The proof in each case will consist in showing, 1st, that the proposition is true of the number one, and 2d, that if true of the number $n$ it is true of the number $1 + n$, next larger than $n$. The different transformations of each expression will be ranged under one another in one column, with the indications of the principles of transformation in another column.

1. To prove the associative principle of addition, that

$$(x + y) + z = x + (y + z)$$

whatever numbers $x$, $y$, and $z$, may be. First it is true for $x = 1$; for

$(1 + y) + z$

$= 1 + (y + z)$ by the definition of addition, 2d clause. Second, if true for $x = n$, it is true for $x = 1 + n$; that is, if $(n + y) + z = n + (y + z)$ then $((1 + n) + y) + z = (1 + n) + (y + z)$. For

$$\begin{aligned}
&((1+n)+y)+z \\
&=(1+(n+y))+z && \text{by the definition of addition:} \\
&=1+((n+y)+z) && \text{by the definition of addition:} \\
&=1+(n+(y+z)) && \text{by hypothesis:} \\
&=(1+n)+(y+z) && \text{by the definition of addition.}
\end{aligned}$$

2. To prove the commutative principle of addition that

$$x+y=y+x$$

whatever numbers $x$ and $y$ may be. First, it is true for $x=1$ and $y=1$, being in that case an explicit identity. Second, if true for $x=n$ and $y=1$, it is true for $x=1+n$ and $y=1$. That is, if $n+1=1+n$, then $(1+n)+1=1+(1+n)$. For $(1+n)+1$

$$\begin{aligned}
&=1+(n+1) && \text{by the associative principle:} \\
&=1+(1+n) && \text{by hypothesis.}
\end{aligned}$$

We have thus proved that, whatever number $x$ may be, $x+1=1+x$, or that $x+y=y+x$ for $y=1$. It is now to be shown that if this be true for $y=n$, it is true for $y=1+n$; that is, that if $x+n=n+x$, then $x+(1+n)=(1+n)+x$. Now,

$$\begin{aligned}
&x+(1+n) \\
&=(x+1)+n && \text{by the associative principle:} \\
&=(1+x)+n && \text{as just seen:} \\
&=1+(x+n) && \text{by the definition of addition:} \\
&=1+(n+x) && \text{by hypothesis:} \\
&=(1+n)+x && \text{by the definition of addition.}
\end{aligned}$$

Thus the proof is complete.

3. To prove the distributive principle, first clause. The distributive principle consists of two propositions:

$$\begin{aligned}
&\text{1st,} && (x+y)\,z=xz+yz \\
&\text{2d,} && x\,(y+z)=xy+xz.
\end{aligned}$$

We now undertake to prove the first of these. First, it is true for $x=1$. For

$$\begin{aligned}
&(1+y)\,z \\
&=z+yz && \text{by the definition of multiplication:} \\
&=1.z+yz && \text{by the definition of multiplication.}
\end{aligned}$$



Second, if true for $x=n$, it is true for $x=1+n$; that is, if $(n+y)\,z=nz+yz$, then $((1+n)+y)\,z=(1+n)\,z+yz$. For

$$\begin{aligned}
&((1+n)+y)\,z \\
&=(1+(n+y))\,z && \text{by the definition of addition:} \\
&=z+(n+y)\,z && \text{by the definition of multiplication:} \\
&=z+(nz+yz) && \text{by hypothesis:} \\
&=(z+nz)+yz && \text{by the associative principle of addition:} \\
&=(1+n)\,z+yz && \text{by the definition of multiplication.}
\end{aligned}$$

4. To prove the second proposition of the distributive principle, that

$$x\,(y+z)=xy+xz.$$

First, it is true for $x=1$; for

$$\begin{aligned}
&1\,(y+z) \\
&=y+z && \text{by the definition of multiplication:} \\
&=1y+1z && \text{by the definition of multiplication.}
\end{aligned}$$

Second, if true for $x=n$, it is true for $x=1+n$; that is, if $n(y+z)=ny+nz$, then $(1+n)\,(y+z)=(1+n)\,y+(1+n)\,z$. For

$$\begin{aligned}
&(1+n)\,(y+z) \\
&=(y+z)+n\,(y+z) && \text{by the definition of multiplication:} \\
&=(y+z)+(ny+nz) && \text{by hypothesis:} \\
&=(y+ny)+(z+nz) && \text{by the principles of addition:} \\
&=(1+n)\,y+(1+n)\,z && \text{by the definition of multiplication.}
\end{aligned}$$

5. To prove the associative principle of multiplication; that is, that

$$(xy)\,z=x\,(yz),$$

whatever numbers $x$, $y$, and $z$, may be. First, it is true for $x=1$, for

$$\begin{aligned}
&(1y)\,z \\
&=yz && \text{by the definition of multiplication:} \\
&=1.yz && \text{by the definition of multiplication.}
\end{aligned}$$

Second, if true for $x=n$, it is true for $x=1+n$; that is, if $(ny)\,z=n\,(yz)$, then $((1+n)\,y)\,z=(1+n)\,(yz)$. For

$$\begin{aligned}
&((1+n)\,y\,)\,z \\
&= (y+ny)\,z \quad \text{by the definition of multiplication:} \\
&= yz + (ny)\,z \quad \text{by the distributive principle:} \\
&= yz + n\,(yz) \quad \text{by hypothesis:} \\
&= (1+n)\,(yz) \quad \text{by the definition of multiplication.}
\end{aligned}$$

6. To prove the commutative principle of multiplication; that

$$xy = yx,$$

whatever numbers $x$ and $y$ may be. In the first place, we prove that it is true for $y = 1$. For this purpose, we first show that it is true for $y = 1$, $x = 1$; and then that if true for $y = 1$, $x = n$, it is true for $y = 1$, $x = 1 + n$. For $y = 1$ and $x = 1$, it is an explicit identity. We have now to show that if $n1 = 1n$ then $(1 + n)\,1 = 1\,(1 + n)$. Now,

$$\begin{aligned}
&(1+n)\,1 \\
&= 1 + n1 \quad \text{by the definition of multiplication:} \\
&= 1 + 1n \quad \text{by hypothesis:} \\
&= 1 + n \quad \text{by the definition of multiplication:} \\
&= 1\,(1+n) \quad \text{by the definition of multiplication.}
\end{aligned}$$

Having thus shown the commutative principle to be true for $y = 1$, we proceed to prove that if it is true for $y = n$, it is true for $y = 1 + n$; that is, if $xn = nx$, then $x\,(1 + n) = (1 + n)\,x$. For

$$\begin{aligned}
&(1+n)\,x \\
&= x + nx \quad \text{by the definition of multiplication:} \\
&= x + xn \quad \text{by hypothesis:} \\
&= 1x + xn \quad \text{by the definition of multiplication:} \\
&= x1 + xn \quad \text{as already seen:} \\
&= x\,(1+n) \quad \text{by the distributive principle.}
\end{aligned}$$

### *Discrete Simple Quantity Infinite in both directions.*

A system of number infinite in both directions has no minimum, but a certain quantity is called *one*, and the numbers as great as this constitute a partial system of semi-infinite number, of which this one is a minimum. The definitions of addition and multiplication require no change, except that the *one* therein is to be understood in the new sense.

$$
\begin{aligned}
&((1 + n)\, y\,)\, z \\
&= (y + ny)\, z && \text{by the definition of multiplication:} \\
&= yz + (ny)\, z && \text{by the distributive principle:} \\
&= yz + n\,(yz) && \text{by hypothesis:} \\
&= (1 + n)\,(yz) && \text{by the definition of multiplication.}
\end{aligned}
$$

6. To prove the commutative principle of multiplication; that

$$xy = yx,$$

whatever numbers $x$ and $y$ may be. In the first place, we prove that it is true for $y = 1$. For this purpose, we first show that it is true for $y = 1$, $x = 1$; and then that if true for $y = 1$, $x = n$, it is true for $y = 1$, $x = 1 + n$. For $y = 1$ and $x = 1$, it is an explicit identity. We have now to show that if $n1 = 1n$ then $(1 + n)\,1 = 1\,(1 + n)$. Now,

$$
\begin{aligned}
&(1 + n)\,1 \\
&= 1 + n1 && \text{by the definition of multiplication:} \\
&= 1 + 1n && \text{by hypothesis:} \\
&= 1 + n && \text{by the definition of multiplication:} \\
&= 1\,(1 + n) && \text{by the definition of multiplication.}
\end{aligned}
$$

Having thus shown the commutative principle to be true for $y = 1$, we proceed to prove that if it is true for $y = n$, it is true for $y = 1 + n$; that is, if $xn = nx$, then $x\,(1 + n) = (1 + n)\,x$. For

$$
\begin{aligned}
&(1 + n)\,x \\
&= x + nx && \text{by the definition of multiplication:} \\
&= x + xn && \text{by hypothesis:} \\
&= 1x + xn && \text{by the definition of multiplication:} \\
&= x1 + xn && \text{as already seen:} \\
&= x\,(1 + n) && \text{by the distributive principle.}
\end{aligned}
$$

*[illegible]tity Infinite in both directions.*

A syste[illegible] in both directions has no minimum, but a certain qua[illegible] and the numbers as great as this constitute a partial system [illegible] number, of which this one is a minimum. The definitions of addition and multiplication require no change, except that the *one* therein is to be understood in the new sense.



To extend the proofs of the principles of addition and multiplication to unlimited number, it is necessary to show that if true for any number $(1 + n)$ they are also true for the next smaller number $n$. For this purpose we can use the same transformations as in the second clauses of the former proof; only we shall have to make use of the following lemma.

If $x + y = x + z$, then $y = z$ whatever numbers $x$, $y$, and $z$, may be. First this is true in case $x = 1$, for then $y$ and $z$ are both next smaller than the same number. Therefore, neither is smaller than the other, otherwise it would not be next smaller to $1 + y = 1 + z$. But in a simple system, of any two different numbers one is smaller. Hence, $y$ and $z$ are equal. Second, if the proposition is true for $x = n$, it is true for $x = 1 + n$. For if $(1 + n) + y = (1 + n) + z$, then by the definition of addition $1 + (n + y) = 1 + (n + z)$; whence it would follow that $n + y = n + z$, and, by hypothesis, that $y = z$. Third, if the proposition is true for $x = 1 + n$, it is true for $x = n$. For if $n + y = n + z$, then $1 + n + y = 1 + n + z$, because the system is simple. The proposition has thus been proved to be true of 1, of every greater and of every smaller number, and therefore to be universally true.

An inspection of the above proofs of the principles of addition and multiplication for semi-infinite number will show that they are readily extended to doubly infinite number by means of the proposition just proved.

The number next smaller than one is called naught, 0. This definition in symbolic form is $1 + 0 = 1$. To prove that $x + 0 = x$, let $x'$ be the number next smaller than $x$. Then,

$$
\begin{aligned}
&x + 0 \\
&= (1 + x') + 0 && \text{by the definition of } x' \\
&= (1 + 0) + x' && \text{by the principles of addition:} \\
&= 1 + x' && \text{by the definition of naught:} \\
&= x && \text{by the definition of } x'.
\end{aligned}
$$

To prove that $x0 = 0$. First, in case $x = 1$, the proposition holds by the definition of multiplication. Next, if true for $x = n$, it is true for $x = 1 + n$. For

$$
\begin{aligned}
&(1 + n)\,0 \\
&= 1\,.\,0 + n\,.\,0 && \text{by the distributive principle:} \\
&= 1\,.\,0 + 0 && \text{by hypothesis:} \\
&= 1\,.\,0 && \text{by the last theorem:} \\
&= 0 && \text{as above.}
\end{aligned}
$$

Third, the proposition, if true for $x = 1 + n$ is true for $x = n$. For, changing the order of the transformations,

$$1 . 0 + 0 = 1 . 0 = 0 = (1 + n) 0 = 1 . 0 + n . 0 .$$

Then by the above lemma, $n . 0 = 0$, so that the proposition is proved.

A number which added to another gives naught is called the negative of the latter. To prove that every number greater than naught has a negative. First, the number next smaller than naught is the negative of one; for, by the definition of addition, one plus this number is naught. Second, if any number $n$ has a negative, then the number next greater than $n$ has for its negative the number next smaller than the negative of $n$. For let $m$ be the number next smaller than the negative of $n$. Then $n + (1 + m) = 0$.

But $n + (1 + m)$
$= (n + 1) + m$ by the associative principle of addition.
$= (1 + n) + m$ by the commutative principle of addition.

So that $(1 + n) + m = 0$. *Q. E. D.* Hence, every number greater than 0 has a negative, and naught is its own negative.

To prove that $(-x)\, y = -(xy)$. We have

$0 = x + (-x)$ by the definition of the negative:
$0 = 0y = (x + (-x))\, y$ by the last proposition but one:
$0 = xy + (-x)\, y$ by the distributive principle:
$-(xy) = (-x)\, y$ by the definition of the negative.

The negative of the negative of a number is that number. For $x + (-x) = 0$. Whence by the definition of the negative $x = -(-x)$.

### *Limited Discrete Simple Quantity.*

Let such a relative term, $c$, that whatever is a $c$ of anything is the only $c$ of that thing, and is a $c$ of that thing only, be called a relative of simple correspondence. In the notation of the logic of relatives,

$$c\breve{c} \prec 1, \quad \breve{c}c \prec 1.$$

If every object, $s$, of a class is in any such relation $c$, with a number of a semi-infinite discrete simple system, and if, further, every number smaller than a number $c$'d by an $s$ is itself $c$'d by an $s$, then the numbers $c$'d by the $s$'s are



said to count them, and the system of correspondence is called a count. In logical notation, putting $g$ for as great as, and $n$ for a positive integral number,

$$s \prec \breve{c}n \quad \breve{g}cs \prec cs.$$

If in any count there is a maximum counting number, the count is said to be finite, and that number is called the number of the count. Let $[s]$ denote the number of a count of the $s$'s, then

$$[s] \prec cs \quad \bar{g}cs \prec \overline{[s]}.$$

The relative "identical with" satisfies the definition of a relative of simple correspondence, and the definition of a count is satisfied by putting "identical with" for $c$, and "positive integral number as small as $x$" for $s$. In this mode of counting, the number of numbers as small as $x$ is $x$.

Suppose that in any count the number of numbers as small as the minimum number, one, is found to be $n$. Then, by the definition of a count, every number as small as $n$ counts a number as small as one. But, by the definition of one there is only one number as small as one. Hence, by the definition of single correspondence, no other number than one counts one. Hence, by the definition of one, no other number than one counts any number as small as one. Hence, by the definition of the count, one is, in every count, the number of numbers as small as one.

If the number of numbers as small as $x$ is in some count $y$, then the number of numbers as small as $y$ is in some count $x$. For if the definition of a simple correspondence is satisfied by the relative $c$, it is equally satisfied by the relative $c$'d by.

Since the number of numbers as small as $x$ is in some count $y$, we have, $c$ being some relative of simple correspondence,

1st. Every number as small as $x$ is $c$'d by a number.

2d. Every number as small as a number that is $c$ of a number as small as $x$ is itself $c$ of a number as small as $x$.

3d. The number $y$ is $c$ of a number as small as $x$.

4th. Whatever is not as great as a number that is $c$ of a number as small as $x$ is not $y$.

Now let $c_1$ be the converse of $c$. Then the converse of $c_1$ is $c$; whence, since $c$ satisfies the definition of a relative of simple correspondence, so also does $c_1$. By the 3d proposition above, every number as small as $y$ is as small as a number that is $c$ of a number as small as $x$. Whence, by the 2d proposition, every number as small as $y$ is $c$ of a number as small as $x$; and it follows that every number as small as $y$ is $c_1$'d by a number. It follows further that every number $c_1$ of a number as small as $y$ is $c_1$ of something $c_1$'d by (that is, $c_1$ being a relative of simple correspondence, is identical with) some number as small as $x$. Also, "as small as" being a transitive relative, every number as small as a number $c$ of a number as small as $y$ is as small as $x$. Now by the 4th proposition $y$ is as great as any number that is $c$ of a number as small as $x$; so that what is not as small as $y$ is not $c$ of a number as small as $x$; whence whatever number is $c$'d by a number not as small as $y$ is not a number as small as $x$. But by the 2d proposition every number as small as $x$ not $c$'d by a number not as small as $y$ is $c$'d by a number as small as $y$. Hence, every number as small as $x$ is $c$'d by a number as small as $y$. Hence, every number as small as a number $c_1$ of a number as small as $y$ is $c_1$ of a number as small as $y$. Moreover, since we have shown that every number as small as $x$ is $c_1$ of a number as small as $y$, the same is true of $x$ itself. Moreover, since we have seen that whatever is $c_1$ of a number as small as $y$ is as small as $x$, it follows that whatever is not as great as a number $c_1$ of a number as small as $y$ is not as great as a number as small as $x$; *i. e.* ("as great as" being a transitive relative) is not as great as $x$, and consequently is not $x$. We have now shown—

1st, that every number as small as $y$ is $c_1$'d by a number;

2d, that every number as small as a number that is $c_1$ of a number as small as $y$ is itself $c_1$ of a number as small as $y$;

3d, that the number $x$ is $c_1$ of a number as small as $y$; and

4th, that whatever is not as great as a number that is $c_1$ of a number as small as $y$ is not $x$.

These four propositions taken together satisfy the definition of the number of numbers as small as $y$ counting up to $x$.

Hence, since the number of numbers as small as one cannot in any count be greater than one, it follows that the number of numbers as small as any number greater than one cannot in any count be one.

Suppose that there is a count in which the number of numbers as small as $1+m$ is found to be $1+n$, since we have just seen that it cannot be 1. In this count, let $m'$ be the number which is $c$ of $1+n$, and let $n'$ be the number which is $c$'d by $1+m$. Let us now consider a relative, $e$, which differs from $c$ only in excluding the relation of $m'$ to $1+n$ as well as the relation of $1+m$ to $n'$



and in including the relation of $m'$ to $n'$. Then $e$ will be a relative of single correspondence; for $c$ is so, and no exclusion of relations from a single correspondence affects this character, while the inclusion of the relation of $m'$ to $n'$ leaves $m'$ the only $e$ of $n'$ and an $e$ of $n'$ only. Moreover, every number as small as $m$ is $e$ of a number, since every number except $1+m$ that is $c$ of anything is $e$ of something, and every number except $1+m$ that is as small as $1+m$ is as small as $m$. Also, every number as small as a number $e$'d by a number is itself $e$'d by a number; for every number $c$'d is $e$'d except $1+m$, and this is greater than any number $e$'d. It follows that $e$ is the basis of a mode of counting by which the numbers as small as $m$ count up to $n$. Thus we have shown that if in any way $1+m$ counts up to $1+n$, then in some way $m$ counts up to $n$. But we have already seen that for $x=1$ the number of numbers as small as $x$ can in no way count up to other than $x$. Whence it follows that the same is true whatever the value of $x$.

If every $S$ is a $P$, and if the $P$'s are a finite lot counting up to a number as small as the number of $S$'s, then every $P$ is an $S$. For if, in counting the $P$'s, we begin with the $S$'s (which are a part of them), and having counted all the $S$'s arrive at the number $n$, there will remain over no $P$'s not $S$'s. For if there were any, the number of $P$'s would count up to more than $n$. From this we deduce the validity of the following mode of inference:

Every Texan kills a Texan,

Nobody is killed by but one person,

Hence, every Texan is killed by a Texan,

supposing Texans to be a finite lot. For, by the first premise, every Texan killed by a Texan is a Texan killer of a Texan. By the second premise, the Texans killed by Texans are as many as the Texan killers of Texans. Whence we conclude that every Texan killer of a Texan is a Texan killed by a Texan, or, by the first premise, every Texan is killed by a Texan. This mode of reasoning is frequent in the theory of numbers.

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NOTE.—It may be remarked that when we reason that a certain proposition, if false of any number, is false of some smaller number, and since there is no number (in a semi-limited system) smaller than every number, the proposition must be true, our reasoning is a mere logical transformation of the reasoning that a proposition, if true for $n$, is true for $1+n$, and that it is true for 1.