No.	Date.	Object.	Magn.	Descr.	Observer.	Cambridge Mean Time.	Remarks.
106	1871 Oct. 21	χ Capricorni	6	D E	C.S.P.	8 19 39.6	
107	1872 Feb. 19	[ω Geminorum]	[6]	D S	C.S.P.	9 27 29.6	
108	1874 Oct. 21	ψ Aquarii	4	D C	R.W.W.	11 55 58.8	
109	Oct. 22	4 Ceti	6	D C	R.W.W.	11 4 44.7	
110	" "	5 Ceti	6	D C	R.W.W.	11 23 55.7	
111	" "	B.A.C. 5	6	D C	R.W.W.	11 48 56.4	
112	Oct. 26	33 Tauri	6	R C	R.W.W.	11 40 40.4	
113	Dec. 19	ϱ^2 Arietis	6	D C	R.W.W.	8 39 49.2	A little late.
114	" "	ϱ^3 Arietis	6	D C	R.W.W.	8 50 29.5	
115	1875 Jan. 20	c Geminorum	6	D C	R.W.W.	6 27 25.3	At cusp; moon low; star unsteady.
116	Mar. 12	......	10	D C	R.W.W.	8 22 23.4	Star's α 3^h 56^m 35^s; δ +23° 47′.
117	" "	36 Tauri	6	D C	R.W.W.	8 32 0.7	
118	" "	......	9	D C	R.W.W.	8 54 31.5	Star's α 3^h 57^m 22^s; δ +24° 7′.
119	Mar. 19	l Leonis	5	D C	R.W.W.	9 45 11.7	
120	April 22	B.A.C. 5253	6	R C	R.W.W.	11 6 17.7	
121*	1876 Aug. 6	Saturn	..	D C	A.S.	11 8 14.3	Bright limb.
122*	" "	"	..	D C	A.S.	11 9 52.8	Bright limb.
123*	" "	"	..	R C	A.S.	11 38 47.9	Dark limb.
124*	" "	"	..	R C	A.S.	11 39 39.8	Dark limb.
125	" "	"	..	R C	A.S.	11 40 6.1	Reappearance of ball. At least 2^s late.
126*	" "	"	..	R C	A.S.	11 41 0.0	
127	" "	"	..	R C	A.S.	11 41 57.7	Last contact of ball. Pretty well ob-[served.
128*	" "	"	..	R C	A.S.	11 42 8.9	
129*	" "	"	..	R C	A.S.	11 42 58.3	
130*	Sept. 2	Saturn	..	D C	A.S.	15 5 25.9	Dark limb.
131	" "	"	..	D C	A.S.	15 5 44.7	First contact of ball.
132	" "	"	..	D C	A.S.	15 6 5.0	Bisection of ball.
133	" "	"	..	D C	A.S.	15 6 27.5	Disappearance of ball.
134	" "	"	..	D C	A.S.	15 6 56.8	Disappearance of ring A.
135*	" "	"	..	R C	A.S.	16 6 29.8	Bisection of ball.
136	" "	"	..	R C	A.S.	16 6 45.1	Last contact of ball.
137	" "	"	..	R C	A.S.	16 6 55.8	Last contact of ring A.

10. Limb wavy; star faint; first distinctly seen $1^s.5$ after recorded time. Power 141.
20. Stars about 1′ apart; northern and fainter first occulted.
26. Perhaps 5^s or 6^s early. Star faint, limb blazing.
46. Well observed; $0^s.5$ too late; this correction has been applied.
49. Well observed; $0^s.3$ too late; this correction has been applied.
60. 2 signals, $11^s.6$ apart; reappearance estimated to precede 1st signal as much as 2d follows it.
67. First seen $11^s.6$ after recorded time; distance from limb doubled $9^s.6$, and tripled $21^s.4$ after first appearance.
121. First contact of ball. Contacts of ring too near those of ball to observe.
122. Disappearance of ball. Too late rather than too early.
123. Reappearance of ring A, outer edge. Not very late.
124. Reappearance of ring B, inner edge. At least 2^s late.
126. Bisection of ball. Too late rather than too early.
128. Last contact of ring B, inner edge. Much too early, perhaps 10^s.
129. Last contact of ring A, outer edge. Pretty well observed.
130. First contact of ring A. Close to terminator; seeing bad.
135. Moon low; planet faint.

Brief Description of the Algebra of Relatives.

BY C. S. PEIRCE.

LET A, B, C, etc., denote objects of any kind. These letters may be conceived to be finite in number or innumerable. The sum of them, each affected by a numerical coefficient (which may equal 0), is called an *absolute term*. Let x be such a term; then we write

$$x = (x)_a A + (x)_b B + (x)_c C + \text{etc.} = \Sigma_i (x)_i I.$$

Here (x), etc., are numbers, which may be permitted to be imaginary or restricted to being real or positive, or to being roots of any given equation, algebraic or transcendental.* By ϕx, any mathematical function of the absolute term x, we mean such an absolute term that

$$(\phi x)_i = \phi (x)_i.$$

That is, each numerical coefficient of ϕx is the function, ϕ, of the corresponding coefficient of x. In particular,

$$(x + y)_i = (x)_i + (y)_i,$$
$$(x \times y)_i = (x)_i \times (y)_i.$$

Otherwise written,

$$x + y = \{(x)_a + (y)_a\} A + \{(x)_b + (y)_b\} B + \text{etc.}$$
$$x \times y = \{(x)_a \times (y)_a\} A + \{(x)_b \times (y)_b\} B + \text{etc.}$$

Two peculiar absolute terms are suggested by the logic of the subject. I call them terms of second intention. The first is zero, 0, and is defined by the equation

$$(0)_i = 0$$

or

$$0 = 0.A + 0.B + 0.C. + \text{etc.}$$

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* I have usually restricted the coefficients to one or other of two values; but the more general view was distinctly recognized in my paper of 1870.

The other is *ens* (or non-relative unity), $\bar{0}$, and is defined by the equation

$$(\bar{0})_i = 1,$$

or

$$\bar{0} = A + B + C + \text{etc.}$$

The symbol $(A:B)$ is called an *individual dual relative*. It signifies simply a pair of individual objects, $(A:B)$ and $(B:A)$ being different. An aggregate of such symbols, each affected by a numerical coefficient, is called a *general dual relative*. The totality of pairs of letters arrange themselves with obvious naturalness in the block,

$A:A$	$A:B$	$A:C$	etc.
$B:A$	$B:B$	$B:C$	etc.
$C:A$	$C:B$	$C:C$	etc.
etc.	etc.	etc.	etc.

If l denotes any general dual relative, then the coefficient of the pair $I:J$ in l is written $(l)_{ij}$. These coefficients are thus each referred to a place in the above block, and may themselves be arranged in the block

$(l)_{aa}$	$(l)_{ab}$	$(l)_{ac}$	etc.
$(l)_{ba}$	$(l)_{bb}$	$(l)_{bc}$	etc.
$(l)_{ca}$	$(l)_{cb}$	$(l)_{cc}$	etc.
etc.	etc.	etc.	etc.

Every relative term, x, is separable into a part called 'self-x,' Sx, such that

$$\mathrm{S}x = \Sigma_i\,(x)_{ii}\,(I:I)$$

and the remaining part, called 'alio-x,' Vx; comprising all the terms in x not in the principal diagonal of the block; so that we write

$$x = \mathrm{S}x + \mathrm{V}x.$$

Each absolute term is considered to be equivalent to a certain relative term; namely,

$$A = (A:A) + (A:B) + (A:C) + \text{etc.}$$

or, if x be an absolute term,

$$(x)_{ij} = (x)_i.$$

The self-part of the relative equivalent to an absolute term is denoted by writing a comma after the term. Accordingly,

$$(x,)_{ii} = (x)_i, \qquad (x,)_{ij} = 0.$$

Besides 0 and $\bar{0}$, two other dual relative terms have been called terms of second intention. These are simply $S\bar{0}$ and $V\bar{0}$. The relative $S\bar{0}$ or $(\bar{0},)$ is also written 1, and is called unity, or 'identical with.' It is defined by the equations

$$(1)_{ii} = 1, \qquad (1)_{ij} = 0.$$

That is,

$$1 = (A:A) + (B:B) + (C:C) + \text{etc.}$$

The relative $V\bar{0}$ is written $\bar{1}$ or $\mathfrak{n}$, and is called 'not,' or 'the negative of.' It is defined by the equations

$$(\bar{1})_{ii} = 0, \qquad (\bar{1})_{ij} = 1.$$

By an absolute function of a relative term is meant that function taken according to the rule for taking the function of an absolute term. That is,

$$(\phi x)_{ij} = \phi\,(x)_{ij}.$$

In particular,

$$(x+y)_{ij} = (x)_{ij} + (y)_{ij}$$
$$(x \times y)_{ij} = (x)_{ij} \times (y)_{ij}$$

Of the various external or relative combinations that have been employed the following may be particularly specified. (1), External multiplication, defined by the equation

$$(x\,y)_{ij} = \Sigma_n\,(x)_{in}\,(y)_{nj}$$

(2), External progressive involution, defined by the equation

$$(x^y)_{ij} = \Pi_n\,(x)_{in}^{(y)_{nj}}$$

(3), External regressive involution, defined by the equation

$$({}^x y)_{ij} = \Pi_n (y)_{nj}^{(x)_{in}}.$$

In general, using Miss Ladd's notation * for the different orders of multiplication,

$$(x \underset{p}{\times} y)_{ij} = \underset{p\text{-}1}{\Pi_n}\{(x)_{in} \underset{p}{\times} (y)_{nj}\}.$$

Other modes of external combination have been used, but they are believed to have only a special utility. Division does not generally yield an unambiguous quotient. Indeed, I have shown that it does so only in the cases of ordinary real algebra, of imaginary algebra, and of real quaternions.

Besides the *mathematical* functions of relatives, there are various modes in

* On De Morgan's Extension of the Algebraic Processes, Am. Jour. Math., Vol. III., No. 3.

which one relative may *logically* depend upon another. Thus, Sx and Vx may be said to be logical functions of x. The most important of such operations is that of taking the *converse* of a relative. The converse of x, written $\breve{x}$ or Kx, is defined by the equation

$$(\breve{x})_{ij} = (x)_{ji}.$$

The algebraical laws of all these combinations are obtained with great facility by a method of which the following are examples: —

Example 1. $\{(xy)z\}_{ij} = \Sigma_n(xy)_{in}(z)_{nj} = \Sigma_n\Sigma_m(x)_{im}(y)_{mn}(z)_{nj}$

$\{x(yz)\}_{ij} = \Sigma_m(x)_{im}(yz)_{mj} = \Sigma_m\Sigma_n(x)_{im}(y)_{mn}(z)_{nj}$

$\therefore\ (xy)z = x(yz).$

Example 2. $\{(x+y)z\}_{ij} = \Sigma_n(x+y)_{in}(z)_{nj} = \Sigma_n\{(x)_{in} + (y)_{in}\}(z)_{nj}$

$= \Sigma_n(x)_{in}(z)_{nj} + \Sigma_n(y)_{in}(z)_{nj} = (xz)_{ij} + (yz)_{ij}$

$\therefore\ (x+y)z = xz + yz.$

The following are some of the elementary formulæ so obtained. Non-relative multiplication is indicated by a comma, relative multiplication by writing the factors one after the other, without the intervention of any sign.

$$(x+y)+z = x+(y+z), \qquad x+y = y+x,$$
$$(x,y),z = x,(y,z), \qquad x,y = y,x,$$
$$(x+y),z = (x,z)+(y,z),$$
$$(xy)z = x(yz),$$
$$(x+y)z = xz+yz, \qquad x(y+z) = xy+xz,$$
$$(x^y)^z = x^{(yz)}, \qquad {}^x({}^y z) = {}^{(xy)}z, \qquad {}^x(y^z) = ({}^x y)^z,$$
$$(x,y)^z = (x^z),(y^z), \qquad {}^x(y,z) = ({}^x y),({}^x z),$$
$$x^{y+z} = (x^y),(x^z), \qquad {}^{x+y}z = ({}^x z),({}^y z),$$
$$kkx = x$$
$$k(x+y) = kx+ky, \qquad k(x,y) = kx,ky$$
$$k(xy) = (ky)(kx), \qquad k(x^y) = {}^{(ky)}(kx)$$
$$0+x = 0, \quad 0,x = 0x = x0 = 0, \quad x^0 = {}^0x = \bar{0},$$
$$\bar{0}\times x = x, \qquad \bar{0}^x = {}^x\bar{0} = \bar{0},$$
$$1x = x1 = x^1 = {}^1x = x,$$
$$({}^x\bar{1})_{ij} = (\bar{1}^x)_{ij} = \begin{cases} 0, \text{ if } x_{ij} \neq 0. \\ 1, \text{ if } x_{ij} = 0. \end{cases}$$

Just as the different pairs of letters, A, B, C, etc., have been conceived to be arranged in a square block, so the different triplets of them may be conceived to be arranged in a cube, and the algebraical sum of all such triplets, each affected with a numerical coefficient, may be called a *triple relative.*

Every dual relative may be regarded as equivalent to a triple relative, just as every absolute term is equivalent to a dual relative.

Every triple relative may be regarded as a sum of five parts, each being a linear expression in terms of one of the five forms,

$$(A:A):A \quad (A:B):A \quad (A:A):B \quad (B:A):A \quad (A:B):C$$

The sign of a dual relative followed by a comma dénotes that part of the equivalent triple relative which consists of terms in one of the forms

$$(A:A):(A:A) \qquad (A:B):(A:B).$$

The multiplication of triple relatives is not perfectly associative and the multiplication of two triple relatives yields a quadruple relative.

The modes of combination of a triple relative followed by two dual relatives are the same as the modes of combination of three dual relatives. This ceases to be true for quadruple and higher relatives.

Corresponding to the operation of taking the converse of a dual relative, there are five operations upon triple relatives. They are defined as follows: —

$$(Ix)_{ijk} = (x)_{jik}, \ (Jx)_{ijk} = (x)_{ikj}, \ (Kx)_{ijk} = (x)_{kji}, \ (Lx)_{ijk} = (x)_{jki}, \ (Mx)_{ijk} = (x)_{kij}.$$

Every quadruple or higher relative may be conceived as a product of triple relatives.

Thus, the essential characteristics of this algebra are (1) that it is a multiple algebra depending upon the addition of square blocks or cubes of numbers, (2) that in the external multiplication the rows of the block of the first factor are respectively multiplied by the columns of the block of the second factor, and (3) that the multiplication so resulting is, for the two-dimensional form of the algebra, always associative. I have proved in a paper presented to the American Academy of Arts and Sciences, May 11, 1875, that this algebra necessarily embraces every associative algebra.

I have here described the algebra apart from the logical interpretation with which it has been clothed. In this interpretation a letter is regarded as a name applicable to one or more objects. By a name is usually meant something representative *of* an object *to* a mind. But I generalize this conception and regard a name as merely something in *a conjoint* relation to a second and a third, that is as a triple relative. A sum of different individual names is a name for each of the things named severally by the aggregant letters. A name multiplied by a positive integral coefficient is the aggregate of so many different senses in which that name may be taken. The individual relative $A : B$ is the name of A considered as the first member of the pair $A : B$. The signification of the external multiplication is then determined by its algebraical definition.

Professor Sylvester, in his "New Universal Multiple Algebra," appears to have come, by a line of approach totally different from mine, upon a system which coincides, in some of its main features, with the Algebra of Relatives, as described in my four papers upon the subject,* and in my lectures on logic. I am unable to judge, from my unprofessional acquaintance with pure mathematics, how much of novelty there may be in my conceptions; but as the researches of the illustrious geometer who has now taken up the subject must draw increased attention to this kind of algebra, I take occasion to redescribe the outlines of my own system, and at the same time to declare my modest conviction that the logical interpretation of it, far from being in any degree special, will be found a powerful instrument for the discovery and demonstration of new algebraical theorems.

BALTIMORE, Jan. 7, 1882.

Postscript.—I have this day had the delight of reading for the first time Professor Cayley's *Memoir on Matrices*, in the *Philosophical Transactions* for 1858. The algebra he there describes seems to me substantially identical with my long subsequent algebra for dual relatives. Many of his results are limited to the very exceptional cases in which division is a determinative process.

My own studies in the subject have been logical not mathematical, being directed toward the essential elements of the algebra, not towards the solution of problems.

JANUARY 16, 1882.

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* Description of a Notation for the Logic of Relatives. Memoirs, American Academy of Arts and Sciences, Vol. IX. 1870. On the Application of Logical Analysis to Multiple Algebra. Proceedings of the same Academy, 1875, May 11. Note on Grassman's Calculus of Extension. Ibid. 1877, Oct. 10. On the Algebra of Logic. Am. Journal of Mathematics, Vol. III.